Particle Filter

Current state-of-the-art data assimilation systems are optimal when

1. Model dynamics are linear
2. observations relate to state variables linearly
3. state variables and observations have Gaussian distributions

For highly nonlinear systems with non-Gaussian distributions, we need to relax the linear, Gaussian assumptions

Recall in Bayesian approach, we assume

$$p(x) = N(\bar{x}^b, P^b), \quad p(y^o|x) = N(y^o, R)$$

in $p(x|y^o) \propto p(y^o|x)\, p(x)$

Now, let's not assume any distribution type and use ensemble model realizations (particles) to sample $p(x)$:

Start with random particles x_k, $k = 1, 2, \cdots, N$

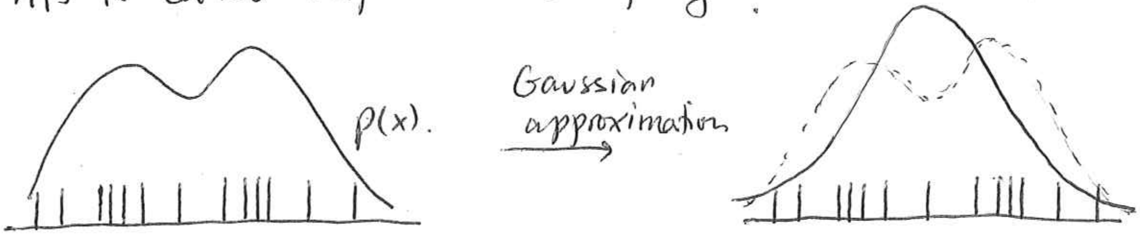
assign equal weights $w_k^b = \frac{1}{N}$ to each particle, so that

$$p(x) = \sum_{k=1}^{N} w_k^b\, \delta(x - x_k)$$

where δ is Dirac's delta function, $\delta(x - x_k) = 0$ if $x \neq x_k$.

and $\int \delta(x - x_k)\, dx = 1$, $\Rightarrow \int p(x)\, dx = \sum_{k=1}^{N} w_k^b = 1$.

→ This is called "importance sampling".



A particle filter generates new weights, w_k^a for each particle, instead of adjusting the particles themselves.

Updated weights are found so that

$$p(x|y^o) \approx \sum_{k=1}^{N} w_k^a \, \delta(x-x_k)$$

$$p(x|y^o) = \frac{p(y^o|x)\,p(x)}{\int p(y^o|x)\,p(x)\,dx} \approx \frac{\sum_{k=1}^{N} p(y^o|x_k)\, w_k^b\, \delta(x-x_k)}{\int \sum_{k=1}^{N} p(y^o|x_k)\, w_k^b\, \delta(x-x_k)\,dx}$$

$$= \sum_{k=1}^{N} \frac{p(y^o|x_k)\, w_k^b}{\sum_{k=1}^{N} p(y^o|x_k)\, w_k^b}\, \delta(x-x_k)$$

$\Rightarrow$ $w_k^a = \dfrac{p(y^o|x_k)\, w_k^b}{\sum_{k=1}^{N} p(y^o|x_k)\, w_k^b}$ are called "importance weights"

The analysis properties of the state, (mean, variance ...) $f(x)$ can be found by $\overline{f(x)} = \int f(x)\,p(x|y^o)\,dx \approx \sum_{k=1}^{N} w_k^a f(x_k)$

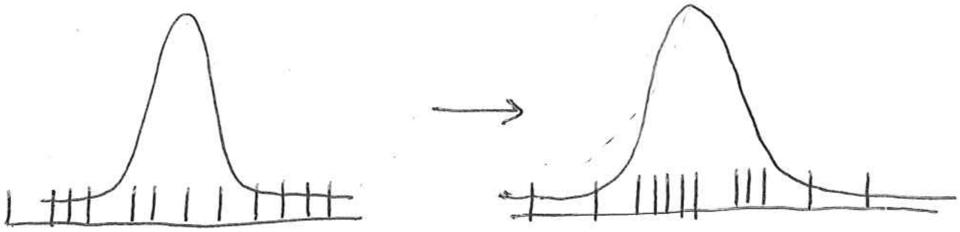
For example, the "analysis mean" $\bar{x} = \int x\, p(x|y^o)\,dx \approx \sum_{k=1}^{N} w_k^a x_k$ is a weighted sum of all particles.

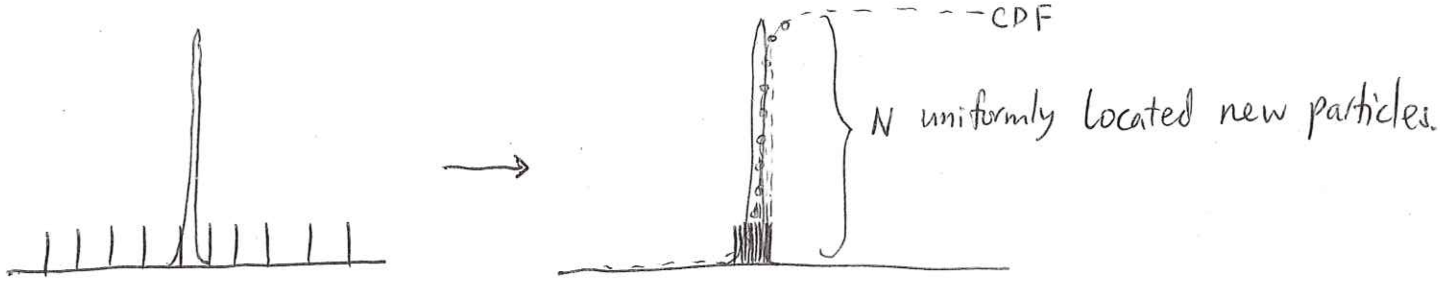
→ More dynamically balanced solution, since more weights are given to similar particles.

→ "filter degeneracy" occurs when importance weights collapse to a single particle. ($w_k^a = 1$ for kth particle and $w_{k\neq l}^a = 0$ for lth particle)

In a "bootstrap filter" introduced by Gordon et al. 1993, a resampling step is used to avoid collapsing of weights.

→ remove trivial particles with close-to-zero weights
duplicate particles with high weights.





problem:

Although particle filter has nice property that as $N \to \infty$, the estimated $p(x|y^o)$ approaches the true Bayesian solution,

$\Rightarrow$ N required to prevent collapse of importance weights to a single particle <u>increases exponentially</u> with the dimension of the system. (Snyder et al. 2008) (Bengtsson et al, 2008) (Bickel et al, 2008)

Local Particle Filter (Poterjoy 2016)

process observations serially, update importance weights using one observation at a time, and limit the impact an observation likelihood has on all weights by a localization function.

for $j = 1, 2, \cdots, p$

$$\vec{\omega}_k^{(j+1)} = \left\{ \left(p(y_j^o \mid \vec{x}_k^{(0)}) - 1 \right) \circ \underbrace{\vec{\rho}_j}_{} + 1 \right\} \circ \vec{\omega}_k^{(j)} \quad , \quad k = 1, 2, \cdots, N$$

schur product

where ρ_j depends on the distance between x_k and y_j^o

updated importance weights:

$$\vec{W}_k^a = \circ \frac{\vec{\omega}_k^{(j+1)}}{\sum_{k=1}^{N} \vec{\omega}_k^{(j+1)}} \quad \leftarrow \text{element-wise division}$$

Comparison between EnKF and PF:

observation y^o — radar reflectivity (dBZ)

state variable x — rain mixing ratio (g/kg)