Adjoint / Ensemble Sensitivity

The tangent linear and adjoint models can help estimate the impact of changes in initial condition ($t=0$) on forecasts (time t)

Consider a scalar metric at forecast time t, which can be expressed as a response function $J(X_t)$. For example, J can be the domain-averaged surface pressure, or kinetic energy, or any other diagnostics.

→ How does changes in initial condition, δx_0, change $J(X_t)$?

A base solution $X_0 \rightarrow X_t = \mathcal{M}(X_0)$

Perturbations around this base trajectory can be found by:

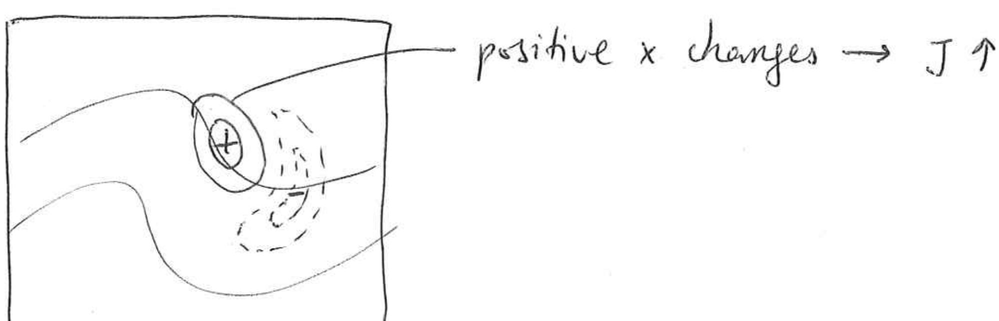
$$\delta X_t = \tilde{M}_t \, \delta X_0 \quad , \quad \text{where } \tilde{M}_t = M_t M_{t-1} \cdots M_2 M_1$$

$$\text{for } \tau = 1, 2, \cdots t \, ; \; M_\tau \equiv \left. \frac{\partial \mathcal{M}}{\partial X} \right|_{X_\tau} = \frac{\partial X_\tau}{\partial X_{\tau-1}}$$

Sensitivity gradient of J with respect to X_t can be expressed as

$$S_t^T = \frac{\partial J}{\partial X_t} = \left(\frac{\partial J}{\partial X_{t,1}} \;\; \frac{\partial J}{\partial X_{t,2}} \;\; \cdots \;\; \frac{\partial J}{\partial X_{t,n}} \right)$$ $1 \times n$ row vector.

Visualize S_t in a map of sensitivity regions with high values indicating location where changes in X contribute more to changes in J



elements in $\tilde{M}_t = \begin{pmatrix} \frac{\partial X_{t,1}}{\partial X_{0,1}} & \frac{\partial X_{t,1}}{\partial X_{0,2}} & \cdots & \frac{\partial X_{t,1}}{\partial X_{0,n}} \\ \frac{\partial X_{t,2}}{\partial X_{0,1}} & \frac{\partial X_{t,2}}{\partial X_{0,2}} & \cdots & \frac{\partial X_{t,2}}{\partial X_{0,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X_{t,n}}{\partial X_{0,1}} & \frac{\partial X_{t,n}}{\partial X_{0,2}} & \cdots & \frac{\partial X_{t,n}}{\partial X_{0,n}} \end{pmatrix}$

To express the sensitivity of $J(X_t)$ to δX_0, we need $\frac{\partial J}{\partial X_0} = S_0^T$

use chain rule: $\delta J = J(X_t + \delta X_t) - J(X_t)$

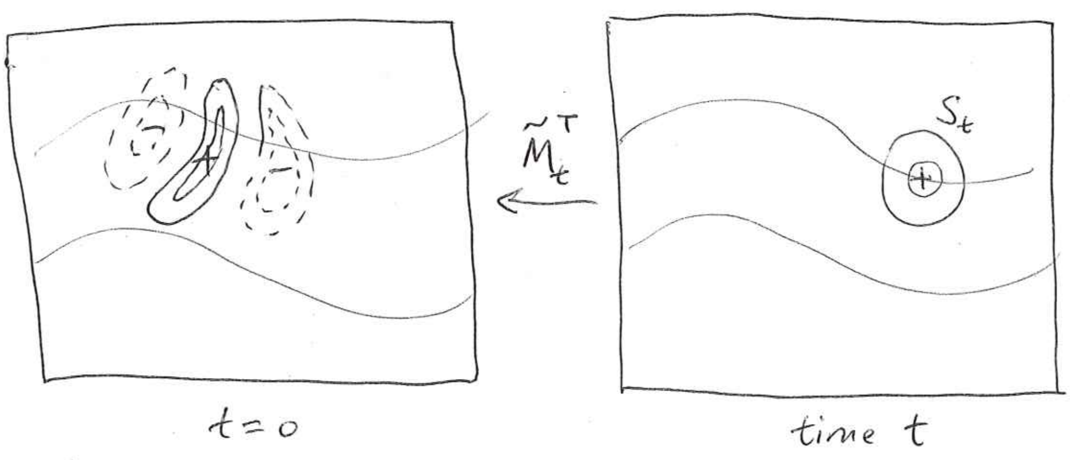
$$\approx \frac{\partial J}{\partial X_t}\delta X_t = \frac{\partial J}{\partial X_t}\frac{\partial X_t}{\partial X_0}\delta X_0$$

$$= \frac{\partial J}{\partial X_t}\tilde{M}_t\,\delta X_0 = S_0^T\,\delta X_0$$

$$\delta J_i = \sum_{l=1}^{n}\sum_{j=1}^{n} \underbrace{\frac{\partial J_i}{\partial X_{t,j}}}_{(S_t)_j}\underbrace{\frac{\partial X_{t,j}}{\partial X_{0,l}}}_{(\tilde{M}_t)_{jl}}\delta X_{0,l}$$

$$S_0 = \left(\frac{\partial J}{\partial X_t}\tilde{M}_t\right)^T = \tilde{M}_t^T S_t$$

use the adjoint model to propagate the Sensitivity gradient calculated at time t back to get S_0,



t=0 time t

use ensemble to estimate $\frac{\partial J}{\partial X_0}$ instead of using adjoint.

for $k=1,2,\cdots,N$, we have member k realization of $X_{0,k} \rightarrow X_{t,k}$ and can calculate $J(X_{t,k}) = \bar{J} + J'_k$, and $X_k = \bar{X} + X'_k$

$$\delta J = \frac{\partial J}{\partial X_0} \delta X_0$$

$$\mathbb{E}(\delta J \, \delta X_0^T) = \frac{\partial J}{\partial X_0} \mathbb{E}(\delta X_0 \, \delta X_0^T)$$

$$\frac{\partial J}{\partial X_0} = \mathbb{E}(\delta J \, \delta X_0^T) \, \mathbb{E}(\delta X_0 \, \delta X_0^T)^{-1} \qquad (1)$$

right hand side can be estimated from ensemble:

$$\mathbb{E}(\delta J \, \delta X_0^T) \approx \frac{1}{N-1} \sum_{k=1}^{N} J'_k \, X'^{T}_{0,k} \qquad (2)$$

$$\mathbb{E}(\delta X_0 \, \delta X_0^T) \approx \frac{1}{N-1} \sum_{k=1}^{N} X'_{0,k} \, X'^{T}_{0,k} \qquad (3)$$

For practical application, sometimes only the diagonal terms in (3) are kept, for easier inversion $\Rightarrow$ ignoring correlation between state variables.