4DVar

Recall from previous lectures for 3DVar:
Bayesian approach finds maximum likelihood estimate of a state variable x given its prior distribution $N(x^b, B)$ and observations $N(y^o, R)$. For Gaussian distributions, the x^a that maximizes posterior probability will minimize the cost function $x^a = \underset{x}{\operatorname{argmin}} J$:

$$J(x) = \frac{1}{2}(x^b - x)^T B^{-1}(x^b - x) + \frac{1}{2}(y^o - h(x))^T R^{-1}(y^o - h(x))$$

In incremental form $\delta x = x^a - x^b$:

$$J(\delta x) = \frac{1}{2}\delta x^T B^{-1}\delta x + \frac{1}{2}(d^{o-b} - H\delta x)^T R^{-1}(d^{o-b} - H\delta x)$$

Up to now, y^o is assumed to contain observations valid at the same time as x. $\rightarrow$ need to include time dimension!

Assume y^o contains observations at a series of discrete time intervals $\tau = 0, 1, 2, \cdots, t$

In order to calculate innovation d^{o-b} at the correct times we first need a model to calculate the trajectory of x over this time window:

for $\tau = 1, 2, \cdots, t$

$$x^b_\tau = \underbrace{m_\tau(x^b_{\tau-1})}_{\text{nonlinear model forecast}}$$

$$d^{o-b}_\tau = y^o_\tau - h_\tau(x^b_\tau), \text{ for } \tau = 0, 1, \cdots, t$$

$\Rightarrow$ replace d^{o-b} in 3DVar with this new (time-matched) d^{o-b}_τ
we call this method 3D-FGAT (first guess at appropriate time).

For time interval $\tau \to \tau+1$, linearize m_τ and h_τ around the background state x^b_τ:

$$\delta x_{\tau+1} \approx M_\tau \, \delta x_\tau \,, \quad M_\tau \equiv \left.\frac{\partial m_\tau}{\partial x}\right|_{x^b_\tau}$$

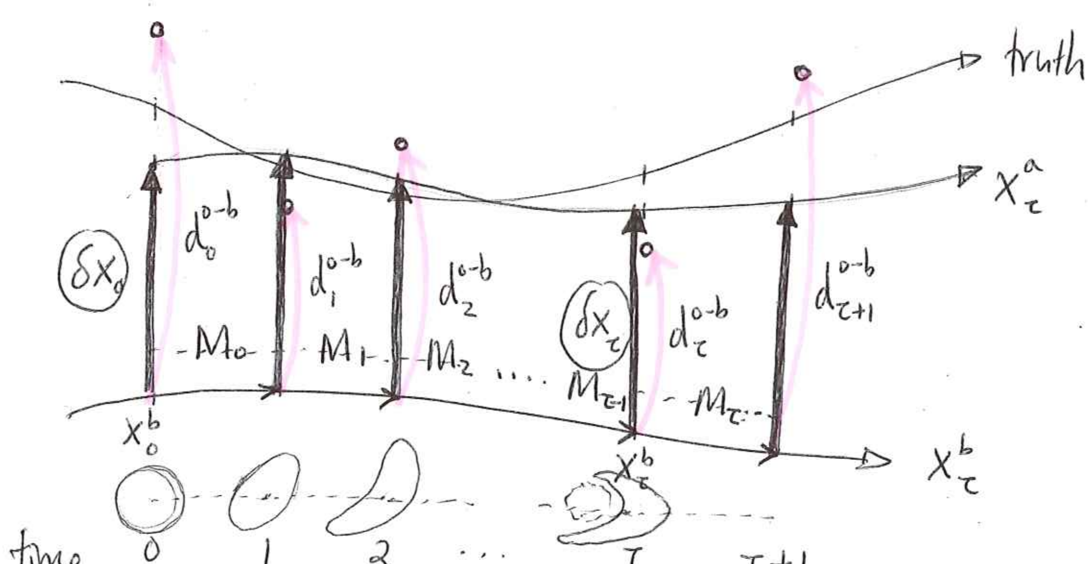
$$\delta y_\tau \approx H_\tau \, \delta x_\tau \,, \quad H_\tau \equiv \left.\frac{\partial h_\tau}{\partial x}\right|_{x^b_\tau}$$

Assume a perfect model, and uncorrelated observation errors. the best estimate of x is found by minimizing the cost function

$$J(\delta x_0) = \frac{1}{2}\delta x_0^T B^{-1} \delta x_0 + \frac{1}{2}\sum_{\tau=0}^{t}\left(d_\tau^{o-b} - H_\tau \tilde{M}_\tau \delta x_0\right)^T R_\tau^{-1}\left(d_\tau^{o-b} - H_\tau \tilde{M}_\tau \delta x_0\right)$$

where $\tilde{M}_{\tau+1} \equiv M_\tau M_{\tau-1} \cdots M_1 M_0$

Analysis $x_0^a = x_0^b + \delta x_0$ is valid at the beginning of the trajectory.



⇒ "Strong constraint" 4DVar

⇒ isotropic B gains some flow-dependency via $\tilde{M}_\tau$.

Consider just the observations at time τ

cost function becomes

$$J(\delta x_0) = \frac{1}{2}\delta x_0^T B^{-1}\delta x_0 + \frac{1}{2}\left(d_\tau^{o-b} - H_\tau \tilde{M}_\tau \delta x_0\right)^T R_\tau^{-1}\left(d_\tau^{o-b} - H_\tau \tilde{M}_\tau \delta x_0\right)$$

solution:

$$\left(B^{-1} + \tilde{M}_\tau^T H_\tau^T R_\tau^{-1} H_\tau \tilde{M}_\tau\right)\delta x_0 = \tilde{M}_\tau^T H_\tau^T R_\tau^{-1} d_\tau^{o-b}$$

$$\delta x_0 = \underbrace{B\tilde{M}_\tau^T H_\tau^T}_{\substack{\text{spatial-temporal}\\ \text{error covariance}}}\left(H_\tau \tilde{M}_\tau B \tilde{M}_\tau^T H_\tau^T + R_\tau\right)^{-1} d_\tau^{o-b}$$

$$= \overline{\varepsilon^b \varepsilon^{bT}}\,\tilde{M}_\tau^T H_\tau^T = \overline{\varepsilon^b \left(H_\tau \tilde{M}_\tau \varepsilon^b\right)^T}$$

To allow the use of imperfect models, we relax the perfect-model assumption, and introduce model error

as $\quad \varepsilon_\tau^m = x_\tau - m_\tau(x_{\tau-1}), \quad \varepsilon_\tau^m \sim N(0, Q_\tau)$

and uncorrelated over time

we can add a penalty term in $J(\delta x_0)$: $\frac{1}{2}\sum_{\tau=1}^{t} \varepsilon_\tau^{mT} Q_\tau^{-1} \varepsilon_\tau^m$

$\Rightarrow$ "Weak constraint" 4DVar

Adding extra constraints:

- Boundary Conditions
- physical balance of solution

$\rightarrow$ Each term in $J(\delta x_0)$ represents a constraint to the final solution, to not let the solution go too far away from something.

Incremental 4DVar :
run outer loop at full model resolution, and perform inner loop cost function minimization at reduced resolution

Work flow of 4DVar:

1. specify $\tilde{B}$ for low resolution.
2. run nonlinear model at full resolution to get x_τ^b

$$x_\tau^b = m_\tau(x_{\tau-1}^b), \quad \text{for } \tau = 1, 2, \cdots, t$$

3. evaluate innovations d_τ^{o-b}

$$d_\tau^{o-b} = y_\tau^o - h_\tau(x_\tau^b), \quad \text{for } \tau = 0, 1, \cdots, t$$

4. Inteppolate x_τ^b to low-resolution $\tilde{x}_\tau^b$
5. Linearize m_τ, h_τ at $\tilde{x}_\tau^b$ to get M_τ, H_τ
6. Define $\delta\tilde{x}_0 = \tilde{x}_0^a - \tilde{x}_0^b$, and preconditioner $LL^T = \tilde{B}$
so control variables $v = L^{-1}\delta\tilde{x}_0$
7. (inner loop).
solve for v by minimizing cost function using CG.

$$\left(I + \sum_{\tau=0}^{t} L^T \tilde{M}_\tau^T H_\tau^T R_\tau^{-1} H_\tau \tilde{M}_\tau L\right) v = \sum_{\tau=0}^{t} L^T \tilde{M}_\tau^T H_\tau^T R_\tau^{-1} d_\tau^{o-b}$$

until $\|\nabla J\|$ small enough, exit
8. $\tilde{x}_0^a = \tilde{x}_0^b + Lv$,
interpolate $\tilde{x}_0^a$ back to full resolution x_0^a
if $\|x_0^a - x_0^b\|$ small enough, return x_0^a as the analysis and exit.
else set $x_0^b = x_0^a$ and goto 2.
(outer loop).