Parameter Estimation

parameters in dynamic model can also be estimated in ensemble data assimilation, given that

- model states are sensitive to the parameters (they are identifiable)
- model with the correct parameters can produce state forecasts that are close to truth.

Although parameters are not directly observed, the choice of parameters influence model state, the closer to true parameters the closer model states will match the observations (truth), thus there are correlations between parameters and observations.

In filtering problem, add parameters θ:

$$x_{t+1} = m(x_t; \theta) + \varepsilon_t^m$$
$$y_{t+1} = h(x_{t+1}) + \varepsilon_{t+1}^o$$

We can augment the state variable to $\begin{pmatrix} x \\ \theta \end{pmatrix}$ to simultaneously update state x and parameters θ,

The filter eqns become:

$$\begin{pmatrix} x^a \\ \theta^a \end{pmatrix} = \begin{pmatrix} x^b \\ \theta^b \end{pmatrix} + P^b H^T (H P^b H^T + R)^{-1} (y^o - h(x^b))$$

where the new observation operator is padded with zeros since θ are not observed.

$$H = \begin{pmatrix} -h_1- & 0 \cdots 0 \\ -h_2- & 0 \cdots 0 \\ \vdots & \\ -h_p- & 0 \cdots 0 \end{pmatrix}$$

Covariance P^b becomes a block matrix $\begin{pmatrix} P^b_{xx} & P^b_{x\theta} \\ P^b_{\theta x} & P^b_{\theta\theta} \end{pmatrix}$

where P^b_{xx} is error covariance for the states

$P^b_{\theta\theta}$ is error covariance for parameters, but since the observations does not directly have θ information, $P_{\theta\theta}$ is actually not needed. $\Rightarrow$ θ update is coming from $P_{\theta x}$, the covariance between θ and x, which propagates y^o observed information about x to θ.

In serial EnKF, using observation y^o_j to update θ:

$$x^{(j+1)} = x^{(j)} + \frac{cov(x^{(j)}, y^{(j)})}{var(y^o_j) + var(y^{(j)})} (y^o_j - y^{(j)})$$

$$\theta^{(j+1)} = \theta^{(j)} + \frac{cov(\theta^{(j)}, y^{(j)})}{var(y^o_j) + var(y^{(j)})} (y^o_j - y^{(j)})$$

where covariances $cov(x^{(j)}, y^{(j)}) = P^{(j)}_{xx} h^T_j = \frac{1}{N-1} \sum_{k=1}^{N} (x'^{(j)}_k \cdot y'^{(j)}_k)$

$$x'^{(j)}_k = x^{(j)}_k - \bar{x}^{(j)}$$

$$y'^{(j)}_k = h_j(x^{(j)}_k) - \overline{h_j(x^{(j)})}$$

$$Cov(\theta^{(j)}, y^{(j)}) = P^{(j)}_{\theta x} h^T_j = \frac{1}{N-1} \sum_{k=1}^{N} (\theta'^{(j)}_k \cdot y'^{(j)}_k)$$

$$\theta'^{(j)}_k = \theta^{(j)}_k - \bar{\theta}^{(j)}$$

ensemble perturbation for θ

$\Rightarrow$ each member is assigned a different $\theta \sim N(\bar{\theta}, P_\theta)$ initially.

Challenges in parameter estimation:

- If parameter is not a constant, but also a complex function of the state: $\theta = \theta(x(t))$
  then the flow dependent change in θ can limit the sample size for the available observations to estimate θ in each x value regime.

- Can the wrong parameters give the best forecast of x?
  If two or more parameters are being estimated, and model configuration (combination of θ values) is not a one-to-one map to final state $\Rightarrow$ It is possible that wrong θ can combine to give good state forecast.
  For example, θ_1 and θ_2 have opposite effect on x's value
  θ_1 too large + θ_2 too small can result in a reasonable x.

- How to maintain the spread of θ over cycles and make sure the ensemble does not collapse on a wrong θ value? $\Rightarrow$ Inflating θ spread.
  For $\theta(x)$ that evolves in time, how to represent its probability $p(\theta \mid x^t, y^o)$?