Ensemble Transform Kalman Filter (ETKF)

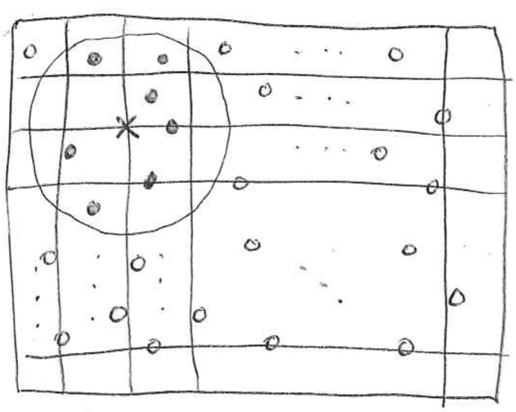
Bishop et al. 2001

Idea: Instead of looping over observations, $j = 1, 2, \cdots, p$ and solve for
$$\vec{x}_{(j+1)} = \vec{x}_{(j)} + \frac{cov(\vec{x}_{(j)}, y_{(j)})}{var(y_j^o) + var(y_{(j)})}(y_j^o - y_{(j)})$$
iteratively (in EnSRF)

we can loop over state variables, $i = 1, 2, \cdots, n$ and solve for
$$x_i^a = x_i^b + \frac{cov(x_i^b, \vec{y}^b)}{var(\vec{y}^o) + var(\vec{y}^b)}(\vec{y}^o - \vec{y}^b)$$

$\Rightarrow$ computation for each grid point i can be done independently, which allows parallelization.

$\Rightarrow$ convert the matrix inversion $(HP^bH^T + R)^{-1}$ into ensemble space to speed up computation.



LETKF (Hunt et al. 2007)
$\Rightarrow$ use only local observations to update each state variable.

Intercomparison of EnKF variants:

Tippett 2003

		Filter Type	Assimilation of obs	Cost
Perturbed obs EnKF	Evensen 1994 Houtekamer Mitchell 1998	stochastic	simultaneous	$O(N^3 + N^2p + N^2n)$
EnSRF	Whitaker Hamill 2002	deterministic	serial	$O(Np + Nnp)$
EAKF	Anderson 2001	deterministic	simultaneous	$O(N^3 + N^2p + N^2n)$
ETKF LETKF	Bishop et al. 2001 Hunt et al. 2007	deterministic	simultaneous	$O(N^3 + N^2p + N^2n)$

The inversion $(HP^bH^T+R)^{-1}$ is a $p\times p$ matrix inversion,
we convert this problem into ensemble space:

Recall ensemble perturbation matrix

$$\mathbf{X}^b = \frac{1}{\sqrt{N-1}}\begin{pmatrix} | & | & & | \\ x_1^{b\prime} & x_2^{b\prime} & \cdots & x_N^{b\prime} \\ | & | & & | \end{pmatrix}, \quad \text{where } x_k^{b\prime} = x_k^b - \bar{x}^b, \text{ for } k=1,2,\cdots,N$$

$$\bar{x}^b = \frac{1}{N}\sum_{k=1}^{N} x_k^b$$

Define $\mathbf{Y}^b \approx H\mathbf{X}^b$

$$= \frac{1}{\sqrt{N-1}}\begin{pmatrix} | & | & & | \\ y_1^{b\prime} & y_2^{b\prime} & \cdots & y_N^{b\prime} \\ | & | & & | \end{pmatrix}, \quad \text{where } y_k^b = h(x_k^b)$$

$$\bar{y}^b = \frac{1}{N}\sum_{k=1}^{N} y_k^b$$

$$y_k^{b\prime} = y_k^b - \bar{y}^b$$

for $k=1,2,\cdots,N$

Kalman Gain can be rewritten:

$$K = P^bH^T(HP^bH^T+R)^{-1} \quad \text{use } P^b = \mathbf{X}^b\mathbf{X}^{bT}$$

$$= \mathbf{X}^b\mathbf{Y}^{bT}(\mathbf{Y}^b\mathbf{Y}^{bT}+R)^{-1} \quad ①$$

$$= \mathbf{X}^b\underbrace{(I+\mathbf{Y}^{bT}R^{-1}\mathbf{Y}^b)^{-1}}_{S}\mathbf{Y}^{bT}R^{-1} \quad ②$$

S is the local analysis error covariance,
similar to $\left((P^b)^{-1}+H^TR^{-1}H\right)^{-1}$ but with background P^b
factored into the second term: $\big(I+\underbrace{(H\mathbf{X}^b)^T}_{\text{"background" error}}\underbrace{R^{-1}}_{\text{"observation" error}}H\mathbf{X}^b\big)^{-1}$

$\longrightarrow$ S is a $N\times N$ matrix, which is much easier to find an inverse:

$$U\Lambda U^T = \mathbf{Y}^{bT}R^{-1}\mathbf{Y}^b$$ is the singular value decomposition of a symmetric matrix.

Note: R is diagonal matrix. so R^{-1} is easy to compute.

$$S = U(I+\Lambda)^{-1}U^T$$

In step ④ of EnKF, we need to generate new ensemble perturbations that satisfy $P^a = (I-KH)P^b$, this can be done by transforming the ensemble perturbation matrix X^b with the square-root of $I-KH$:

$$
\begin{aligned}
P^a = X^a X^{aT} &= (I-KH)P^b \\
&= \left[I - P^b H^T (H P^b H^T + R)^{-1} H\right] X^b X^{bT} \\
&= X^b X^{bT} - X^b Y^{bT} (Y^b Y^{bT} + R)^{-1} Y^b X^{bT} \\
&= X^b \underbrace{\left(I - Y^{bT} (Y^b Y^{bT} + R)^{-1} Y^b\right)}_{③} X^{bT} \\
&= X^b \underbrace{\left(I + Y^{bT} R^{-1} Y^b\right)^{-1}}_{④} X^{bT} \\
&= X^b S X^{bT}
\end{aligned}
$$

It is much easier to find the square root of $S = WW^T$ than $\left(I - Y^{bT}(Y^b Y^{bT} + R)^{-1} Y^b\right)$

$$W = U(I+\Lambda)^{-\frac{1}{2}}$$

$$X^a = X^b W$$

Proof that ③ = ④:

$$
\begin{aligned}
③④^{-1} &= \left[I - y^T (y y^T + R)^{-1} y\right]\left[I + y^T R^{-1} y\right] \qquad yy^T + R - R \\
&= I + y^T R^{-1} y - y^T (yy^T + R)^{-1} y - y^T (yy^T+R)^{-1} yy^T R^{-1} y \\
&= I + \boxed{y^T R^{-1} y} - y^T (yy^T + R)^{-1} y - \boxed{y^T \cancel{(yy^T+R)^{-1}} \cancel{(yy^T+R)} R^{-1} y} \\
&\qquad + y^T (yy^T+R)^{-1} \cancel{R R^{-1}} y \\
&= I
\end{aligned}
$$

$$
\begin{aligned}
② = ④\, y^T R^{-1} = ③\, y^T R^{-1} &= y^T R^{-1} - y^T (yy^T + R)^{-1} y y^T R^{-1} \\
&= y^T R^{-1} - y^T \cancel{(yy^T+R)^{-1}} \cancel{(yy^T+R)} R^{-1} + y^T (yy^T+R)^{-1} \cancel{R R^{-1}} \\
&= ①
\end{aligned}
$$

LETKF algorithm:

(1) Forecast step $x^b_{t+1,k} = m(x^a_{t,k})$ for $k = 1, 2, \cdots, N$

(2)-(8) Analysis step at time t+1. independently performed for each $i = 1, 2, \cdots, n$

(2) Compute ensemble pertubations $X^b_i = \frac{1}{\sqrt{N-1}}(x^{b\prime}_{1,i} \; x^{b\prime}_{2,i} \cdots x^{b\prime}_{N,i})$

(3) compute observation prior perturbations Y^b by applying nonlinear observation operator h.

(4) Localization

At each grid point i, select local subset of observations (observations within cutoff radius)

$y^o_L = \text{subset}(y^o)$, $\bar{y}^b_L = \text{subset}(\bar{y}^b)$,

$Y^b_L = \text{subset}(Y^b)$

$R_L = \text{subset}(R) \circ \text{diag}(\rho_L)$, R-localization function $(\rho_L)_j = e^{d_{ij}^2/2L^2}$

where d_{ij} is distance between ith grid point and j-th observation.

L is the localization length scale

→ 3.65L is considered the cutoff distance.

(5) Compute local analysis error covariance in ensemble space.

$$S = (I + Y^{bT}_L R^{-1}_L Y^b_L)^{-1}$$

(6) update mean: $\bar{x}^a_i = \bar{x}^b_i + X^b_i w$

$$w = S Y^{bT}_L R^{-1}_L (y^o_L - \bar{y}^b_L)$$

(7) update ensemble perturbations $X^a_i = X^b_i W$, $W = S^{\frac{1}{2}}$

(8) form analysis $x^a_{k,i} = \bar{x}^a_i + x^{a\prime}_{k,i}$ for $k = 1, 2, \cdots, N$

→ analysis is local linear combination of ensemble perturbations, which is limited by the directions defined by ensemble.