Innovation Statistics

Desroziers et al. 2005

A set of diagnostics in observation space.

Recall $\vec{d}^{o-b} \equiv \vec{y}^o - \vec{h}(\bar{x}^b)$ (1)

Define another two differences:

$$\vec{d}^{o-a} \equiv \vec{y}^o - \vec{h}(\bar{x}^a) \quad (2)$$

$$\vec{d}^{a-b} \equiv \vec{h}(\bar{x}^a) - \vec{h}(\bar{x}^b) \quad (3)$$

Note that $\overline{h(x)}$ is used instead of $h(\bar{x})$ for some cases.

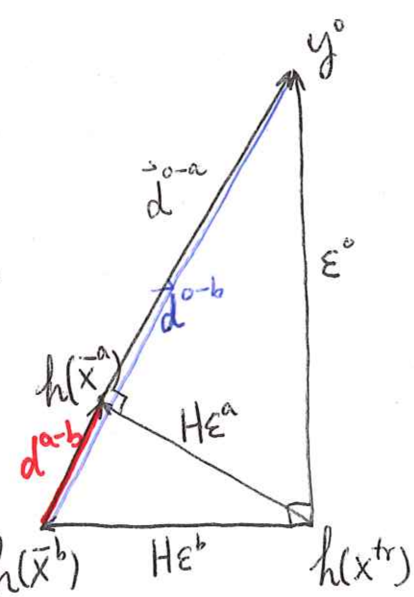
$\vec{y}^o = \vec{h}(x^{tr}) + \vec{\varepsilon}^o$

$\bar{x}^b = x^{tr} + \varepsilon^b$

$h(\bar{x}^b) = h(x^{tr}) + H\varepsilon^b$

Similarly, $h(\bar{x}^a) = h(x^{tr}) + H\varepsilon^a$

→ Geometric representation of relation between $\vec{d}^{o-b}$, $\vec{d}^{a-b}$ and $\vec{d}^{o-a}$



→ $\vec{\varepsilon}^o$ is orthogonal to $\vec{H\varepsilon^b}$, because observation errors are not correlated with background errors:

$$E(\varepsilon^o \varepsilon^{bT}) = 0$$

$$d^{o-b} = h(x^{tr}) + \varepsilon^o - h(x^{tr}) - H\varepsilon^b$$

$$= \varepsilon^o - H\varepsilon^b \quad (4)$$

$$\mathbb{E}\left(d^{o-b}(d^{o-b})^T\right) = \mathbb{E}\left(\varepsilon^o\varepsilon^{oT}\right) + H\,\mathbb{E}\left(\varepsilon^b\varepsilon^{bT}\right)H^T$$
$$= R + HP^bH^T \qquad (5)$$

- This relation should hold if filter is performing as expected: P^b is correctly characterizing background error ε^b.
- If P^b is too small compared to actual $\mathbb{E}(\varepsilon^b\varepsilon^{bT})$, which means filter is too confident about the background, after assimilation, $P^a = (I-KH)P^b$ is reduced even more. Over time, when P^b is always too small, less and less weight will be given to observation, eventually ignoring observations
$$x^a = (I-KH)x^b + K\,y^o$$
$$K \to 0\,,\quad x^a \to x^b$$

This is called "catastrophic filter divergence"
→ filter solution diverge from truth, yet P^b keeps decreasing.

→ One way to diagnose for this situation is calculating the "Consistency ratio":
$$CR = \sqrt{\frac{\mathrm{tr}\left[HP^bH^T + R\right]}{\mathrm{tr}\left[\mathbb{E}\left(d^{o-b}(d^{o-b})^T\right)\right]}}$$
actual spread

expected spread

If $CR = 1$: filter performance is well characterized by P^b → good spread skill.

If $CR < 1$: ensemble is under-dispersive
→ susceptible to filter divergence

If $CR > 1$: ensemble is over-dispersive.

Several other equalities:

$$d^{a-b} \cong H(\bar{x}^a - \bar{x}^b) = HK d^{o-b}$$

$$\begin{aligned}
\mathbb{E}\left[d^{a-b}(d^{o-b})^T\right] &= HK\,\mathbb{E}\left[d^{o-b}(d^{o-b})^T\right] \\
&= HP^bH^T\cancel{(HP^bH^T+R)^{-1}}\,\cancel{(HP^bH^T+R)} \\
&= HP^bH^T
\end{aligned}$$

$$\begin{aligned}
d^{o-a} &= y^o - h(\bar{x}^b) + h(\bar{x}^b) - h(\bar{x}^a) \\
&\cong d^{o-b} - H(\bar{x}^a - \bar{x}^b) = (I - HK)\,d^{o-b}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[d^{o-a}(d^{o-b})^T\right] &= (I - HK)\,\mathbb{E}\left[d^{o-b}(d^{o-b})^T\right] \\
&= R\,(HP^bH^T+R)^{-1}(HP^bH^T+R) \\
&= R
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[d^{a-b}(d^{o-a})^T\right] &= HK\,\mathbb{E}\left[d^{o-b}(d^{o-b})^T\right](I-HK)^T \\
&= HP^bH^T(HP^bH^T+R)^{-1}R = H(I-KH)P^bH^T \\
&= HP^aH^T
\end{aligned}$$

In summary:

$$\mathbb{E}\left[d^{o-b}(d^{o-b})^T\right] = HP^bH^T + R$$

$$\mathbb{E}\left[d^{a-b}(d^{o-b})^T\right] = HP^bH^T$$

$$\mathbb{E}\left[d^{o-a}(d^{o-b})^T\right] = R$$

$$\mathbb{E}\left[d^{a-b}(d^{o-a})^T\right] = HP^aH^T$$

Hint: use the triangle to remember these.