Square Root Modification

Whitaker and Hamill 2002

In practice, we don't perturb observations for each member

In the update step of serial EnKF:

for $k = 1, 2, \cdots, N$

$$x_{(j+1),k} = x_{(j),k} + K_j (y^o_{j,k} - y_{(j),k}) \qquad (1)$$

$\bar{x}$ and x' are updated separately instead:

$$\bar{x}_{(j+1)} = \bar{x}_{(j)} + K_j (\underbrace{y^o_j}_{\text{unperturbed observation}} - \bar{y}_{(j)}) \qquad (2)$$

for $k = 1, 2, \cdots, N$

$$\begin{aligned} x'_{(j+1),k} &= x'_{(j),k} + K_j (0 - y'_{(j),k}) \qquad (3) \\ &= (I - K_j h_j) x'_{(j),k} \end{aligned}$$

Missing observation perturbations will cause $P^{(j+1)}$ to be erroneous.

The correct $P^{(j+1)} = (I - K_j h_j) P^{(j)}$ according to Kalman filter.

The actual $P^{(j+1)}$ according to (3) is

$$\begin{aligned} P^{(j+1)} &= \frac{1}{N-1} \sum_{k=1}^{N} x'_{(j+1),k} \, x'^{T}_{(j+1),k} \\ &= (I - K_j h_j) \frac{1}{N-1} \sum_{k=1}^{N} x'_{(j),k} \, x'^{T}_{(j),k} (I - K_j h_j)^T \\ &= (I - K_j h_j) P^{(j)} (I - K_j h_j)^T \end{aligned}$$

$\rightarrow$ an extra $(I - K_j h_j)$ factor cause $P^{(j+1)}$ to be too small.

Note: the correct $P^{(j+1)} = (I - K_j h_j)\, P^{(j)}$

scalar

$$= (I - K_j h_j)\, P^{(j)} (I - K_j h_j)^T + K_j R_{jj} K_j^T$$

To reconcile, add a "square root modification" factor ϕ for the K_j in update eqns for x'

(3) becomes for $k = 1, 2, \cdots, N$

$$x'_{(j+1),k} = (I - \phi K_j h_j)\, x'_{(j),k} \qquad (4)$$

So that

$$P^{(j+1)} = \underbrace{(I - \phi K_j h_j)\, P^{(j)} (I - \phi K_j h_j)^T}_{(5)} = \underbrace{(I - K_j h_j)\, P^{(j)}}_{(6)}$$

Solve for ϕ:

scalar

$$(5) = P^{(j)} - \phi K_j h_j P^{(j)} - \phi P^{(j)} h_j^T K_j^T + \phi^2 K_j (h_j P^{(j)} h_j^T) K_j^T$$

can show that $P^{(j)} h_j^T K_j^T = K_j h_j P^{(j)}$

$$\underline{K_j (h_j P^{(j)} h_j^T + R_{jj}) K_j^T} = \underline{P^{(j)}} h_j^T K_j^T$$

$$= K_j \underline{h_j P^{(j)}} \qquad (7)$$

since $K_j (\underbrace{h_j P^{(j)} h_j^T}_{} + \underbrace{R_{jj}}_{}) = \underbrace{P^{(j)}}_{} h_j^T$

symmetric matrices

$$(5) = P^{(j)} - 2\phi K_j h_j P^{(j)} + \phi^2 K_j h_j P^{(j)} h_j^T K_j^T$$

$\parallel$

$$(6) = P^{(j)} - K_j h_j P^{(j)}$$

$$\phi^2 K_j h_j P^{(j)} h_j^T K_j^T - (2\phi - 1)\, \underline{K_j h_j P^{(j)}} = 0$$

use (7): $$K_j\left(\phi^2 h_j P^{(j)} h_j^T\right) K_j^T - (2\phi-1)\, \underline{K_j\left(h_j P^{(j)} h_j^T + R_{jj}\right) K_j^T} = 0$$

$$\phi^2 \underbrace{h_j P^{(j)} h_j^T}_{\text{scalar}} - (2\phi-1)\left(\underbrace{h_j P^{(j)} h_j^T + R_{jj}}_{\text{scalar}}\right) = 0$$

$$\frac{h_j P^{(j)} h_j^T}{h_j P^{(j)} h_j^T + R_{jj}}\, \phi^2 - 2\phi + 1 = 0$$

Define as a

$$a\left(\phi^2 - \frac{2}{a}\phi + \frac{1}{a^2} - \frac{1}{a^2} + \frac{1}{a}\right) = 0$$

$$\left(\phi - \frac{1}{a}\right)^2 = \frac{1}{a^2} - \frac{1}{a} = \frac{1-a}{a^2}$$

$$\phi = \frac{1}{a} \pm \frac{\sqrt{1-a}}{a}$$

choose solution within $(0,1)$: $\phi = \dfrac{1-\sqrt{1-a}}{a} = \dfrac{1}{1+\sqrt{1-a}}$

$$\phi = \left(1 + \sqrt{\frac{R_{jj}}{h_j P^{(j)} h_j^T + R_{jj}}}\right)^{-1}$$

Note: $R_{jj} = var(y_j^o)$

$h_j P^{(j)} h_j^T = var(y_{(j)})$

ϕ is calculated for each observation (j).