Ensemble Kalman Filter (EnKF)

Evensen 1994

- EnKF is a Monte Carlo approximation of the Extended KF.
- Instead of propagating P_t^a forward using M and M^T, use an ensemble to sample $N(\bar{x}_{t+1}^b, P_{t+1}^b)$ with the nonlinear model.

Fundamental Steps:

step forward in time

① forecast step, run ensemble of nonlinear models from $N(x_t^a, P_t^a)$

② Determine $\bar{x}_{t+1}^b$ and P_{t+1}^b → ensemble mean and pertubations.

③ Update $\bar{x}_{t+1}^b, P_{t+1}^b \to \bar{x}_{t+1}^a, P_{t+1}^a$ using Kalman filter eqns.

④ Generate new ensemble pertubations, satisfying P_{t+1}^a

Notation:

$i = 1, 2, \cdots, n$ state variables (if subscript not used, indicating the whole vector).

$j = 1, 2, \cdots, p$ observations

$k = 1, 2, \cdots, N$ ensemble, $\bar{x}$ ensemble mean

x_k' ensemble pertubation for kth member

x_t state at time t, (usually t subscripts are omitted here)

x^b, x^a: prior (background), posterior (analysis)

(vector signs $\vec{\ }$ are omitted too), Step ① Ensemble forecast , $x^a_{t,k} \sim N(\bar{x}^a_t, P^a_t)$

for $k = 1, 2, \cdots, N$

$$x^b_{t+1,k} = m(x^a_{t,k})$$

there is no random model error ε^m_t here

→ using deterministic nonlinear forecast model.

Step ②

$$\bar{x}^b_{t+1} = \frac{1}{N} \sum_{k=1}^{N} x^b_{t+1,k}$$

Define an ensemble perturbation matrix

$$\underline{X}^b = \frac{1}{\sqrt{N-1}} \begin{pmatrix} | & | & & | \\ x_1' & x_2' & \cdots & x_N' \\ | & | & & | \end{pmatrix}^b \qquad (\text{t+1 subscript omitted})$$

where $x^{b\prime}_k = x^b_k - \bar{x}^b$

Recall the (i_1, i_2)th element in P^b :

$$P^b_{i_1 i_2} = \overline{\varepsilon^b_{i_1} \varepsilon^b_{i_2}} \cong \frac{1}{N-1} \sum_{k=1}^{N} (x^b_{i_1,k} - \bar{x}^b_{i_1})(x^b_{i_2,k} - \bar{x}^b_{i_2})$$

$$\therefore \; P^b \cong \underline{X}^b \underline{X}^{bT}$$

Note: → P^b is a sample-estimated error covariance : Since each member samples a nonlinear model trajectory, the estimated P^b contains flow-dependent structures.

→ Average of the ensemble $\bar{x}^b$ does not always provide better solution for a model ; sometimes features are displaced among members, taking the average will smooth out these features. → use medium instead ?

Step (3) Analysis step

Same as Extended Kalman filter, for ensemble mean:

$$\bar{x}^a = \bar{x}^b + K(y^o - h(\bar{x}^b)) \qquad (1)$$

$$K = P^b H^T (H P^b H^T + R)^{-1} \qquad (2)$$

for error covariance:

$$P^a = (I - KH) P^b \qquad (3)$$

$\longrightarrow$ how to let analysis ensemble satisfy this relation? $\to$ step (4)

step (4) is not unique, there are several ways to do this. (different "flavors" of EnKF).

In Evensen 1994, and later: Houtekamer (Canada):

- update members so that P^a matches this relation.
- also, create observation perturbations $y^{o\prime} \sim N(0, R)$ for the update of ensemble perturbations $\Rightarrow$ "perturbed observation" EnKF

$x_k^a = \bar{x}^a + x_k^{a\prime}$ is the analysis member k.

update: $$x_k^{a\prime} = x_k^{b\prime} + K(y_k^{o\prime} - H x_k^{b\prime}) \quad \text{for } k = 1, 2, \cdots, N \qquad (4).$$

Note: Consider $x^{a\prime}$, $x^{b\prime}$ and $y^{o\prime}$ as realizations (random draws) of ε^a, ε^b and ε^o, the update eqns for $x_k^{a\prime}$ should result in $\mathbb{E}(\varepsilon^a \varepsilon^{aT}) = P^a$ that satisfies (3).

- without the need for $P_{t+1}^b = M P_t^a M^T + Q$, the EnKF is feasible for large-dimensional systems!
- can combine (1) and (4) into one update eqn for each member.

Schematic of EnKF work flow: