Kalman Filter

Consider a Linear Gaussian system x,

$$x_{t+1} = M x_t + \varepsilon_t^m , \quad \varepsilon_t^m \sim N(0, Q) \quad \text{is the dynamic system.}$$

$$y_{t+1}^o = H x_{t+1} + \varepsilon_{t+1}^o , \quad \varepsilon_{t+1}^o \sim N(0, R) \quad \text{is the observation}$$

Goal; at time t, given unbiased estimate of $x \sim N(\bar{x}_t^a, P_t^a)$
find the best linear unbiased estimate (BLUE) for x
a time $t+1$, $N(\bar{x}_{t+1}^a, P_{t+1}^a)$

$$\bar{x}_{t+1}^a = V_{t+1} \bar{x}_t^a + W_{t+1} y_{t+1}^o \quad (\text{linear combination})$$

$$\varepsilon_{t+1}^a = \bar{x}_{t+1}^a - x_{t+1}$$

$$= V_{t+1} \bar{x}_t^a + W_{t+1}\Big(H\big(M x_t + \varepsilon_t^m\big) + \varepsilon_{t+1}^o\Big) - \big(M x_t + \varepsilon_t^m\big)$$

$\bar{x}_t^a = x_t + \varepsilon_t^a$, unbiased $\mathbb{E}(\varepsilon_t^a) = 0$

we want $\mathbb{E}(\varepsilon_{t+1}^a) = 0$:

$$\varepsilon_{t+1}^a = V_{t+1} \varepsilon_t^a + V_{t+1} x_t + \big(W_{t+1} HM - M\big) x_t + \big(W_{t+1} H - I\big)\varepsilon_t^m + W_{t+1} \varepsilon_{t+1}^o$$

$$\mathbb{E}(\varepsilon_{t+1}^a) = \mathbb{E}\Big[\big(V_{t+1} + W_{t+1} HM - M\big) x_t\Big]$$

Since $\mathbb{E}(\varepsilon_t^a) = 0$, $\mathbb{E}(\varepsilon_t^m) = 0$, $\mathbb{E}(\varepsilon_{t+1}^o) = 0$.

$$\therefore \quad V_{t+1} = \big(I - W_{t+1} H\big) M$$

define $\bar{x}_{t+1}^b = M \bar{x}_t^a$

$$\bar{x}_{t+1}^a = \big(I - W_{t+1} H\big) M \bar{x}_t^a + W_{t+1} y_{t+1}$$

$$= \bar{x}_{t+1}^b + W_{t+1}\big(y_{t+1}^o - H \bar{x}_{t+1}^b\big)$$

$$\mathbb{E}(\varepsilon_t^a \varepsilon_t^{aT}) = P_t^a$$

$$\varepsilon_{t+1}^b = \bar{x}_{t+1}^b - x_{t+1} = M\bar{x}_t^a - (Mx_t + \varepsilon_t^m) = M\varepsilon_t^a + \varepsilon_t^m$$

$$P_{t+1}^b = \mathbb{E}(\varepsilon_{t+1}^b \varepsilon_{t+1}^{bT}) = MP_t^a M^T + Q \quad , \text{ assuming } \mathbb{E}(\varepsilon_t^a \varepsilon_t^m) = 0$$

We want the <u>best</u> estimate, find W_{t+1} so that sum of analysis variances is minimum.

$$\varepsilon_{t+1}^a = (I - W_{t+1}H)\,\varepsilon_{t+1}^b + W_{t+1}\varepsilon_{t+1}^o \quad , \text{ assuming } \mathbb{E}(\varepsilon_{t+1}^b \varepsilon_{t+1}^o) = 0$$

$$\mathbb{E}(\varepsilon_{t+1}^a \varepsilon_{t+1}^{aT}) = P_{t+1}^a = (I - W_{t+1}H)P_{t+1}^b (I - W_{t+1}H)^T + W_{t+1}RW_{t+1}^T$$

$$\frac{\partial\, tr(P_{t+1}^a)}{\partial W_{t+1}} = 0$$

If B is symmetric

$$\frac{\partial\, tr(ABA^T)}{\partial A} = 2AB$$

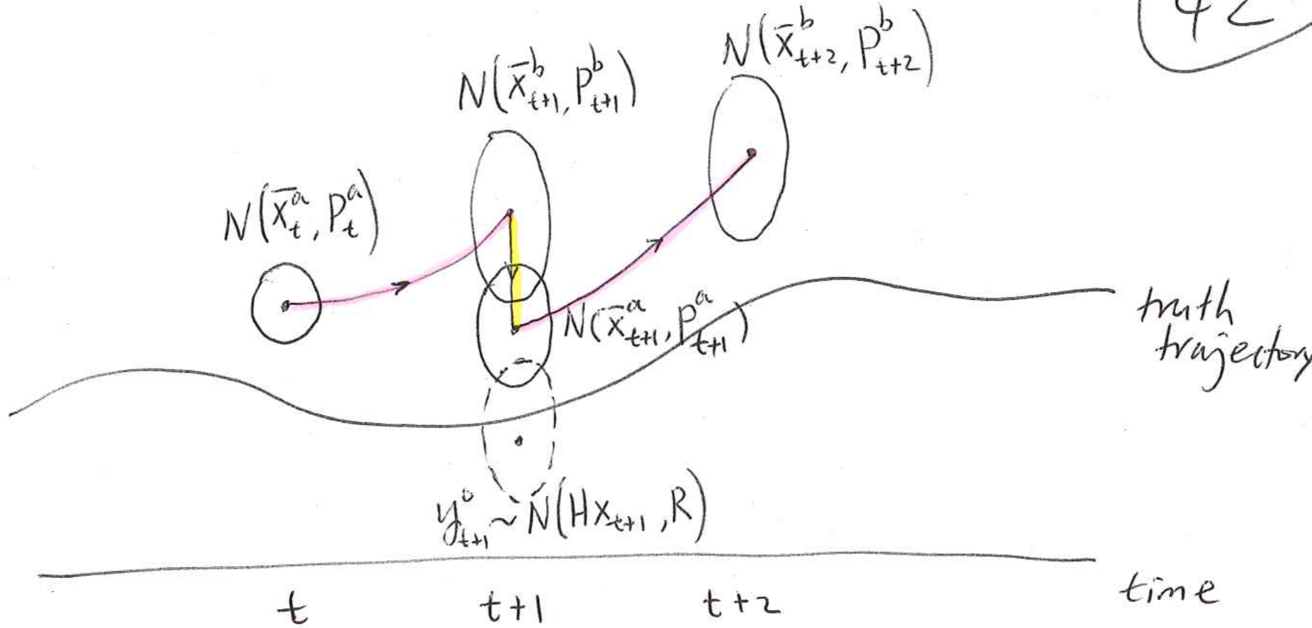
$$\frac{\partial\, tr(AC)}{\partial A} = C^T$$

$$= -2P_{t+1}^b H^T + 2W_{t+1}HP_{t+1}^b H^T + 2W_{t+1}R$$

$$W_{t+1} = P_{t+1}^b H^T (HP_{t+1}^b H^T + R)^{-1} = K_{t+1}$$

is the "Kalman Gain" matrix.

→ similar to W in OI, but B is replaced with a covariance that evolves over time, P_{t+1}^b



Given posterior estimate of the state at time t $N(\bar{x}^a_t, P^a_t)$

1. Forecast step.

   Get background (prior) estimate for time t+1.

$$\bar{x}^b_{t+1} = M\bar{x}^a_t \qquad (1)$$

$$P^b_{t+1} = MP^a_t M^T + Q \qquad (2)$$

2. Analysis step,

   Get posterior estimate for time t+1

$$\bar{x}^a_{t+1} = \bar{x}^b_{t+1} + K_{t+1}(y^o_{t+1} - H\bar{x}^b_{t+1}) \qquad (3)$$

$$K_{t+1} = P^b_{t+1}H^T(HP^b_{t+1}H^T + R)^{-1} \qquad (4)$$

$$P^a_{t+1} = (I - K_{t+1}H)P^b_{t+1}(I - K_{t+1}H)^T + K_{t+1}RK^T_{t+1}$$

$$= \cdots = (I - K_{t+1}H)P^b_{t+1} \qquad (5)$$

(Kalman 1960; Kalman, Bucy 1961)

Derivation for (5)

$$(I-KH)P^b(I-KH)^T + KRK^T$$

$$= (I-KH)P^b \underbrace{- (I-KH)P^b(KH)^T + KRK^T}$$

replace $K = P^bH^T(HP^bH^T+R)^{-1}$

$K(HP^bH^T+R) = P^bH^T$

$$= -P^bH^TK^T + KHP^bH^TK^T + KRK^T$$

$$= -P^bH^TK^T + K(HP^bH^T+R)K^T$$

$$= -P^bH^TK^T + P^bH^TK^T = 0$$

→ (1)-(5) are the Kalman filter eqns for a linear Gaussian system.

$N(\bar{x}, P)$ evolves overtime,

$\bar{x}^a$ is the BLUE, in terms of minimum error variances.

→ Note that update of P is offline (not dependent on $\bar{x}$)