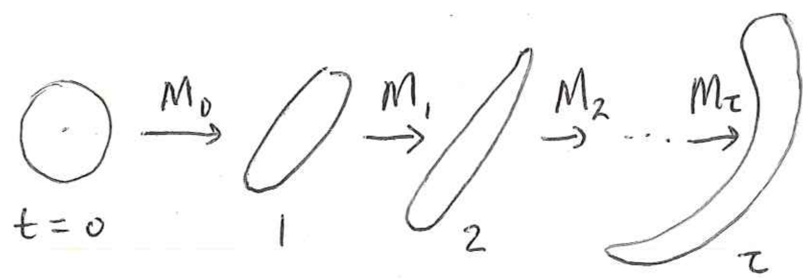
Characterize Error Growth

TLM contains information about the dynamic system

$$\delta \vec{X}_{t+1} = \underset{t \to t+1}{M} \delta \vec{X}_t$$

one can derive TLM for time $t = 0, 1, 2, \cdots, \tau$
and use them to study the evolution of initial error.

$$\delta \vec{x}_\tau = \tilde{M} \delta \vec{x}_0 = M_\tau M_{\tau-1} \cdots M_1 M_0 \delta \vec{x}_0$$

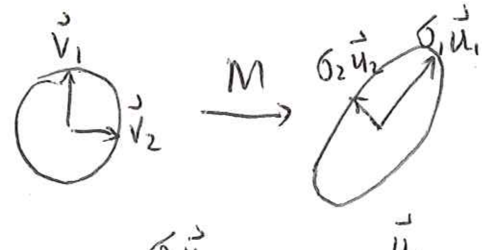


To determine the fastest-growing error mode:
perform singular value decomposition (SVD) on M:

$$U^T M V = S = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \ \sigma_n \end{pmatrix}, \quad U^T U = V^T V = I$$

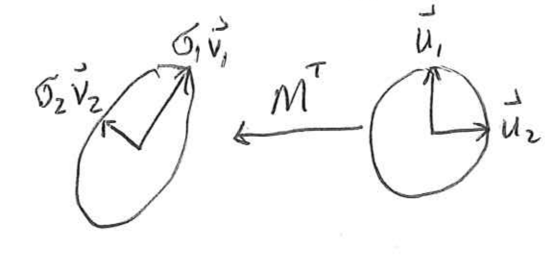
- $MV = US$

  $M\vec{v}_i = \sigma_i \vec{u}_i$



- $M^T U = VS$

  $M^T \vec{u}_i = \sigma_i \vec{v}_i$



$\vec{u}_i$ are called "singular vectors"

Note: singular values σ_i can also be found by eigenvalue decomposition of MM^T:

$$MM^T U = USS$$

$$MM^T \vec{u}_i = \sigma_i^2 \vec{u}_i$$

- Lyapunov Vectors

- singular value σ_i describes the stretching of $\vec{u}_i$ direction over a finite time interval τ. (local)

  The long-term linear growth

$$\lambda_i = \lim_{\tau \to \infty} \frac{1}{\tau} \ln[\sigma_i(t_0+\tau)] \quad , \quad (\sigma_i = e^{\lambda_i t})$$

  is called Lyapunov exponents. (global)

- During a long integration, the growth rate converges to the Leading Lyapunov exponent

$$\lambda_1 = \lim_{\tau \to \infty} \frac{1}{\tau} \ln\left[\frac{\|\vec{u}_{t_0+\tau}\|}{\|\vec{u}_{t_0}\|}\right]$$

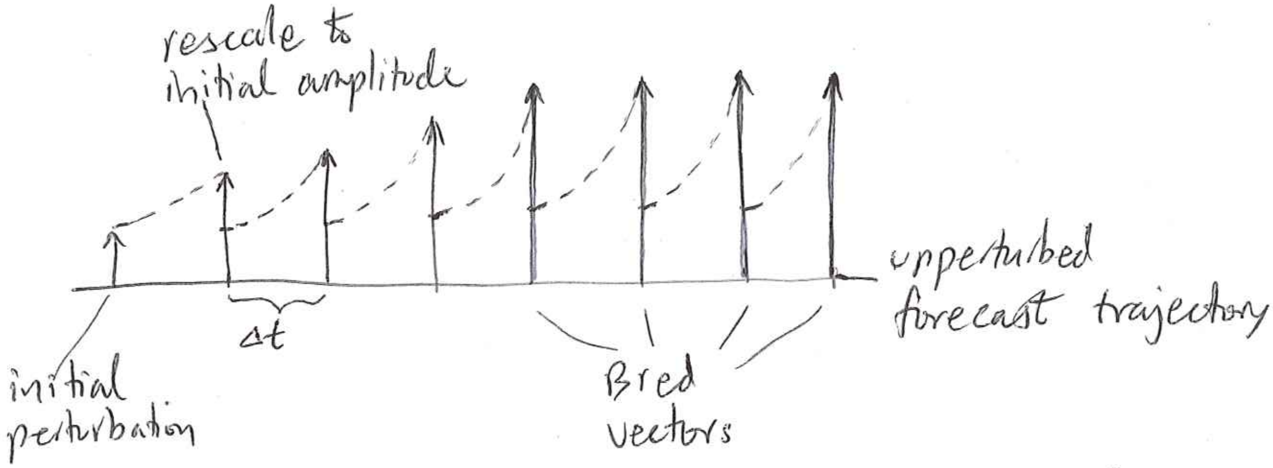
  the Leading Lyapunov Vector (LLV)

$$\vec{l}_1 = \lim_{\tau \to \infty} M_\tau \cdots M_1 \vec{u}_t$$

Problem: MM^T can be ill-conditioned for large system (large n), consider using ensemble methods, such as breeding, to find fast error-growth modes.
(power method)

Bred vector

Similar to Lyapunov Vectors, but using the nonlinear model to integrate for a long time. (Toth, Kalnay 1993)



- tunable parameters: rescaling interval and amplitude.
- Local growth rate $= \frac{1}{\Delta t} \ln\left(\frac{\|\delta x_t\|}{\|\delta x_{t-1}\|}\right)$
- Bred vector dimension

$$\Psi(\sigma_1, \sigma_2, \cdots \sigma_n) = \left(\sum_{i=1}^{n} \sigma_i\right)^2 \Big/ \sum_{i=1}^{n} \sigma_i^2$$

is the local effective dimension of the local bred vector subspace.