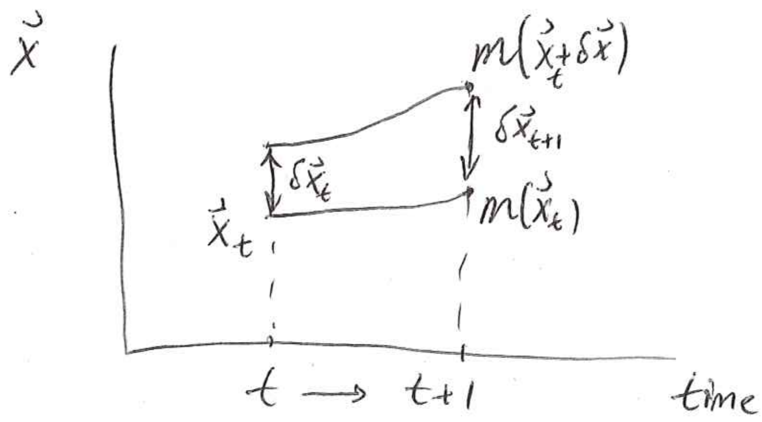
Tangent Linear Model

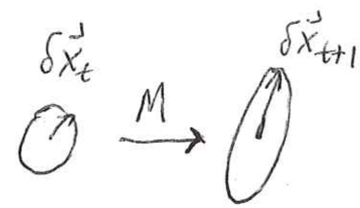
A dynamic model (dynamical system) has its own error growth mechanism and predictability. By studying a linearized version of the system, we can characterize how errors evolve over time.

Let the nonlinear dynamical system be denoted as

$$\vec{x}_{t+1} = m(\vec{x}_t) = \vec{x}_t + \Delta t\, f(\vec{x}_t) \quad \text{(Euler forward)}.$$

when there is a small deviation $\delta \vec{x}_t$ at time t the model trajectory from $\vec{x}_t + \delta\vec{x}_t$ is different from that starting from $\vec{x}_t$:





The difference of state at time t+1 can be approximated

$$m(\vec{x}_t + \delta\vec{x}_t) = m(\vec{x}_t) + \left.\frac{\partial m}{\partial \vec{x}}\right|_{\vec{x}_t} \delta\vec{x}_t + \cdots$$

$$\delta\vec{x}_{t+1} = m(\vec{x}_t + \delta\vec{x}_t) - m(\vec{x}_t) \cong \underset{t\to t+1}{M}\, \delta\vec{x}_t\,, \qquad \underset{t\to t+1}{M} \equiv \left.\frac{\partial m}{\partial \vec{x}}\right|_{\vec{x}_t} = \nabla_{\vec{x}}\, m(\vec{x}_t)$$

- M propagates a perturbation <u>forward</u> in time.

M is called "Tangent Linear" model (TLM)
it is also the Jacobian matrix $M_{ij} = \frac{\partial m_i}{\partial x_j}$

Example 1: find the TLM for L63:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\sigma x_1 + \sigma x_2 \\ \rho x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - \beta x_3 \end{pmatrix} \qquad \frac{d\vec{x}}{dt} = f(\vec{x})$$

in discrete form using Euler forward. $\vec{x}_{t+1} = \vec{x}_t + \Delta t \cdot f(\vec{x}_t)$
$= m(\vec{x}_t)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{t+1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_t + \Delta t \begin{pmatrix} -\sigma x_1 + \sigma x_2 \\ \rho x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - \beta x_3 \end{pmatrix}_t = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

Linearize: $\frac{\partial m_i}{\partial x_j}$

$$M = \begin{pmatrix} \frac{\partial m_1}{\partial x_1} & \frac{\partial m_1}{\partial x_2} & \frac{\partial m_1}{\partial x_3} \\ \frac{\partial m_2}{\partial x_1} & \frac{\partial m_2}{\partial x_2} & \frac{\partial m_2}{\partial x_3} \\ \frac{\partial m_3}{\partial x_1} & \frac{\partial m_3}{\partial x_2} & \frac{\partial m_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 + \Delta t(-\sigma) & \Delta t \sigma & 0 \\ \Delta t(\rho - x_3) & 1 - \Delta t & -\Delta t x_1 \\ \Delta t x_2 & \Delta t x_1 & 1 - \Delta t \beta \end{pmatrix}$$

$$= I + \Delta t \cdot \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - x_3 & -1 & -x_1 \\ x_2 & x_1 & -\beta \end{pmatrix}_t = I + \Delta t F$$

$$\delta\vec{x}_{t+1} = M\delta\vec{x}_t = \delta\vec{x}_t + \underbrace{\Delta t \cdot F \cdot \delta\vec{x}_t}$$

change in perturbation
due to model forcing

Example 2: TLM for L96

Discrete form using Euler forward:

$$x_i^{t+1} = m(x_i^t) = x_i^t + \Delta t\left[(x_{i+1}^t - x_{i-2}^t)\,x_{i-1}^t - x_i^t + F\right]$$

$$\delta\vec{x}_{t+1} = M\,\delta\vec{x}_t = \underbrace{\left(I + \Delta t\,F^*\right)}_{\text{TLM}}\delta\vec{x}_t$$

ADM

$$M\,\delta\vec{x}_t = \delta\vec{x}_t + \Delta t \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & -1 & X_{i-3} & 0 & 0 & \cdots \\ \cdots & X_i - X_{i-3} & -1 & X_{i-2} & 0 & \cdots \\ \cdots & -X_{i-1} & X_{i+1} - X_{i-2} & -1 & X_{i-1} & \cdots \\ \cdots & 0 & -X_i & X_{i+2} - X_{i-1} & -1 & \cdots \\ \cdots & 0 & 0 & -X_{i+1} & X_{i+3} - X_i & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}_t \begin{array}{c} \\ \\ \\ \text{TLM} \\ \\ \\ \\ \end{array} \begin{bmatrix} \vdots \\ \delta X_{i-2} \\ \delta X_{i-1} \\ \delta X_i \\ \delta X_{i+1} \\ \delta X_{i+2} \\ \vdots \end{bmatrix}_t$$

- nonlinear L96 code:

for $i = 1, 2, \cdots, n$

$$x_i^{t+1} = x_i^t + \Delta t\left((x_{i+1}^t - x_{i-2}^t)\,x_{i-1}^t - x_i^t + F\right)$$

- TLM L96 code:

for $i = 1, 2, \cdots, n$

$$\delta x_i^{t+1} = (1 - \Delta t)\,\delta x_i^t + \Delta t\left[-x_{i-1}^t\,\delta x_{i-2}^t + (x_{i+1}^t - x_{i-2}^t)\,\delta x_{i-1}^t + x_{i-1}^t\,\delta x_{i+1}^t\right]$$

- ADM L96 code:

for $i = 1, 2, \cdots, n$

$$\delta x_i^{t*} = (1 - \Delta t)\,\delta x_i^{t+1} + \Delta t\left[x_{i-2}^{t+1}\,\delta x_{i-1}^{t+1} + (x_{i+2}^{t+1} - x_{i-1}^{t+1})\,\delta x_{i+1}^{t+1} - x_{i+1}^{t+1}\,\delta x_{i+2}^{t+1}\right]$$