Nonlinear Test Models

Test models for developing data assimilation schemes:
- simplified, efficient to compute
- capture the key dynamical process of interest, for which there are challenges in performing data assimilation.

Lorenz 1963 model: (L63)

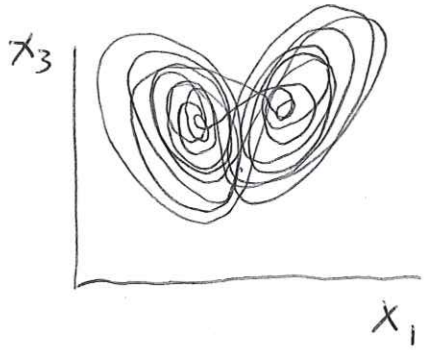
$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

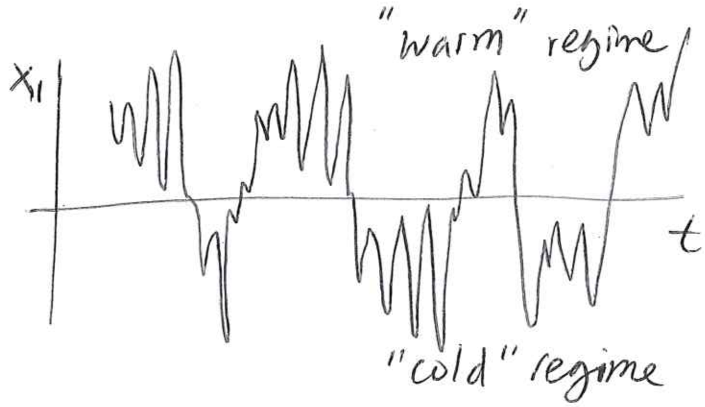
$$\begin{cases} \dfrac{dx_1}{dt} = \sigma(x_2 - x_1), \\ \dfrac{dx_2}{dt} = x_1(\rho - x_3) - x_2, \\ \dfrac{dx_3}{dt} = x_1 x_2 - \beta x_3 \end{cases}$$

→ an approximation of Navier-Stokes equations, which describes convective movement in the atmosphere.

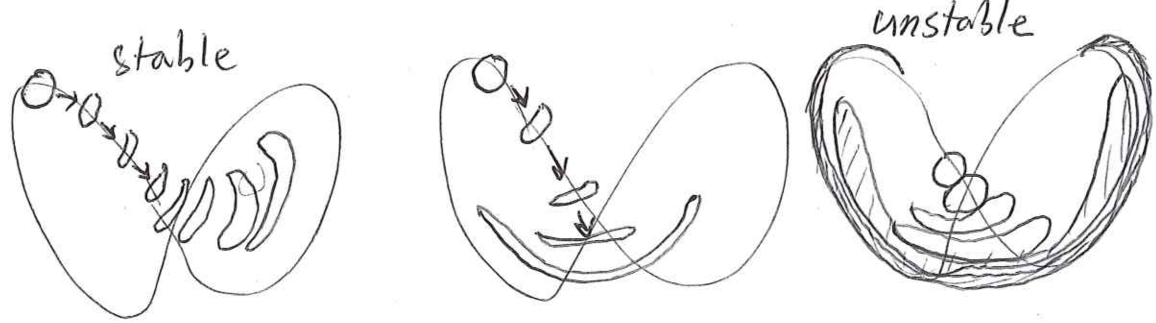
→ a typical configuration: $\sigma = 10$, $\beta = 8/3$, $\rho = 28$
displays "chaotic" behavior: sensitive to initial condition

Solution phase space:





chaotic behavior: (Palmer et al. 2007)



- consider an initial volume of uncertain conditions the trajectories of points inside the volume end up in a new volume after some integration time, and the volume changes shape.
- Somewhere in the phase space (left most one). features relatively stable trajectories, with volumes slowly stretching. "errors grow slowly"; while somewhere (rightmost one) features very rapid error growth and bifurcation of the volume into two regimes.

- sensitivity to initial condition → predictability

- Linear error growth : stretching into a hyper ellipsoid
  nonlinear " " phase: folding is needed for the solution to stay within the bounds.

  → asymptote to a "strange attractor"

Lorenz 1996 model (L96)

Defined on a 1D cyclic domain with n grid points:

$i = 1, 2, \cdots, n$

$$\frac{dX_i}{dt} = \underbrace{(X_{i+1} - X_{i-2}) X_{i-1}}_{\text{similar to nonlinear advection}} - \underbrace{X_i}_{\text{dissipation}} + \underbrace{F}_{\text{constant "solar" forcing}}$$

similar to nonlinear advection $\sim u \frac{\partial u}{\partial x}$

dissipation $\nu \nabla^2 u \sim -u$

constant "solar" forcing

→ typical choice $n = 40$, $F = 8$

→ the cyclic domain mimics the mid-latitude belt of 32000 km with grid spacing 800 km, which is about the Rossby deformation radius.

→ Solution features "Rossby-like" waves with unstable westward phase velocity, collectively move eastward.

→ 0.2 time unit ~ 1 day in real atmosphere

- conserves energy norm: $E = \sum_{i=1}^{n} x_i^2 / 2$

$$\sum_{i=1}^{n} x_i \frac{dx_i}{dt} = \sum_{i=1}^{n} \left[x_i x_{i-1}(x_{i+1} - x_{i-2}) - x_i^2 + F x_i \right]$$

$$\frac{dE}{dt} = -E + F \sum_{i=1}^{n} x_i / 2 = 0$$

- solar forcing scale ~ 1 day, dissipation ~ 5 days. Decorrelation scale ~ 3 days.

How to numerically integrate a model?

- See "Numerical Method" courses for more details.

Discretization in space and <u>time</u>. (L63, L96 already discretized in space).

<u>Euler forward in time:</u>

for a model $\frac{d\vec{x}}{dt} = f(\vec{x})$

$$\vec{x}^{t+1} = \vec{x}^{t} + \Delta t \cdot f(\vec{x}^{t}) \quad \text{(explicit scheme)}.$$

<u>Runge-Kutta 4th-order scheme:</u> (RK4).

$$\vec{x}_{(1)} = \vec{x}^{t} + \frac{\Delta t}{2} f(\vec{x}^{t})$$

$$\vec{x}_{(2)} = \vec{x}^{t} + \frac{\Delta t}{2} f(\vec{x}_{(1)})$$

$$\vec{x}_{(3)} = \vec{x}^{t} + \Delta t\, f(\vec{x}_{(2)})$$

$$\vec{x}^{t+1} = \vec{x}^{t} + \Delta t \left[\frac{f(\vec{x}^{t})}{6} + \frac{f(\vec{x}_{(1)})}{3} + \frac{f(\vec{x}_{(2)})}{3} + \frac{f(\vec{x}_{(3)})}{6} \right]$$

- Euler forward requires shorter Δt to achieve numerical stability than RK4.

<u>Implicit scheme</u>, e.g. Crank-Nicholson

$$\vec{x}^{t+1} = \vec{x}^{t} + \Delta t\, f\left(\frac{\vec{x}^{t} + \vec{x}^{t+1}}{2}\right)$$

→ since $\vec{x}^{t+1}$ is on both left and right hand side. need to solve matrix inversion!

→ numerically stable!