Preconditioning

- We can speed up the convergence of the minimization by a change of variables $v = L^{-1}\delta x$, v is called Control variable

- L is chosen so that the new cost function has a more spherical Hessian $\nabla^2 J$

$$L = B^{\frac{1}{2}}, \quad B = LL^T$$

$$J_b(v) = \frac{1}{2}\delta x^T B^{-1}\delta x = \frac{1}{2}(Lv)^T B^{-1} Lv = \frac{1}{2}v^T L^T (LL^T)^{-1} Lv$$

$$= \frac{1}{2}v^T v$$

$$J_o(v) = \frac{1}{2}(\underbrace{y^o - h(x^b)}_{d^{o-b}} - H\delta x)^T R^{-1}(d^{o-b} - H\delta x)$$

$$= \frac{1}{2}(d^{o-b} - HLv)^T R^{-1}(d^{o-b} - HLv)$$

- Hessian becomes: $I + L^T H^T R^{-1} HL$, easier to calculate
- the presence of I guarantee that the minimum eigenvalue is ≥ 1, there is no small eigenvalues to destroy the conditioning of the problem.

$$\nabla_v J(v) = v + L^T H^T R^{-1}(HLv - d^{o-b}) = 0$$

after finding v solution, convert back to $\delta x = Lv$.

$$x^a = x^b + \delta x$$

More on $B = \overline{\varepsilon_b \varepsilon_b^T}$:

$B_{ij} = \overline{\varepsilon_{bi} \varepsilon_{bj}} = \sigma_i \sigma_j \rho_{ij}$

the covariance matrix can be modelled as

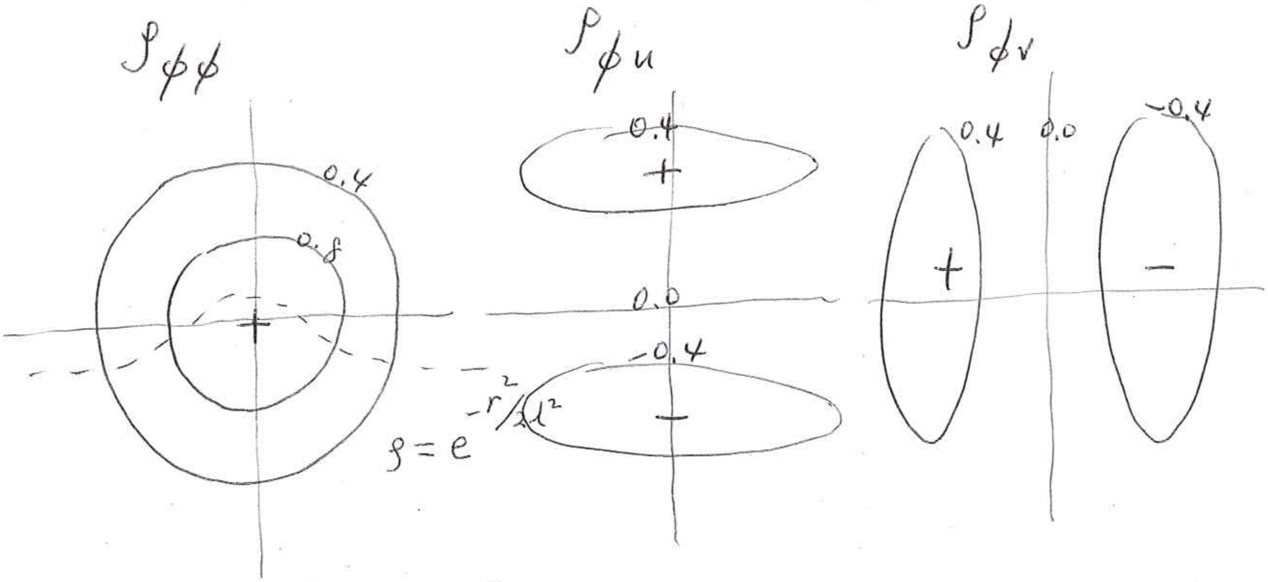
$$B = DCD$$

where $D = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \sigma_n \end{pmatrix}$, and $C = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{pmatrix}$ is the correlation matrix.

(D) is the standard deviation diagonal matrix

correlations: determine the shape of analysis increment.

For weather models, assume increments in u, v, ϕ are related through geostrophic balance: $\frac{\partial \phi}{\partial y} = -fu$, $\frac{\partial \phi}{\partial x} = fv$



Assumptions in B

- isotropic
- separable in horizontal and vertical
- geostrophic balance (what about tropics?)
- constant in time

Note: analysis increment can only occur in the subspace spanned by B:

- Assume $B = bb^T$, so that background error can take place only in b direction.

- from OI update equation:

$$\delta x = BH^T(HBH^T + R)^{-1} d^{o-b}$$
$$= b \underbrace{\boxed{(Hb)^T (HBH^T + R)^{-1} d^{o-b}}}_{\text{a scalar}}$$

- the analysis increment can only take place along b direction too!

, More ideally, the space spanned by B should comprise several directions that analysis increments can follow. (determined by the dynamics)?

$$B = LL^T = \sum_i b_i b_i^T \quad , \quad L = (b_1 \; b_2 \cdots b_n).$$

Finding the preconditioner $L = B^{\frac{1}{2}}$ using special case of singular value decomposition for symmetric B:

$$B = UDU^T \quad , \quad \begin{array}{l} U \text{ is orthogonal matrix} \\ D \text{ is diagonal} \;\; " \end{array}$$

$$L = UD^{\frac{1}{2}}U^T \quad \text{so that} \quad LL^T = UD^{\frac{1}{2}} \overbrace{U^T U}^{=I} D^{\frac{1}{2}} U^T$$
$$= UDU^T = B$$

Putting everything together, 3DVar algorithm:

- Specify B by Gaussian covariance, or NMC method.
- x^b from model forecast (forecast step).
- y^o from observations in a time window
- (analysis step): solve for $\delta x = x^a - x^b$

outer loop

Linearize observation operator H at x^b

calculate $d^{o-b} = y^o - h(x^b)$

preconditioning: $v = L^{-1}\delta x$

$$\underbrace{\left(I + L^T H^T R^{-1} H L\right)}_{A'} v = \underbrace{L^T H^T R^{-1} d^{o-b}}_{b'}$$

evaluate A' and b', solve for v using a minimizer (e.g. CG) ← inner loop.

$x^a = x^b + Lv$

- if x^a converges to a solution, start next cycle.