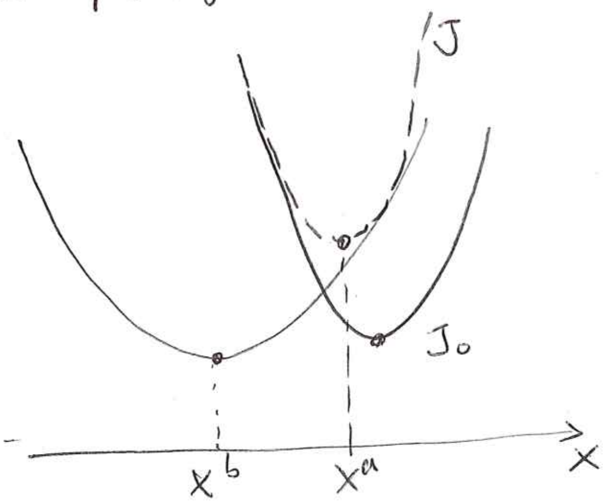
3DVar

Variational method solves the same optimization problem of finding x^a using x^b and y^o and their uncertainties B and R. It uses a cost function formulation instead.

$$J(x) = -\ln(p(x|y^o))$$
$$= -\ln(p(y^o|x)\,p(x)) + c$$
$$= \underbrace{\tfrac{1}{2}(x-x^b)^T B^{-1}(x-x^b)}_{J_b} + \underbrace{\tfrac{1}{2}(y^o-h(x))^T R^{-1}(y^o-h(x))}_{J_o} + c$$

J_b fit to background | J_o fit to observation

solution $x = x^a$ is found by minimizing $J(x)$ so that $p(x|y^o)$ is maximized → x^a is most probable to be x^t



For a quadratic function $F(x) = \frac{1}{2}x^T A x + d^T x + c$ has gradient $\nabla_x F = Ax + d$, when A is symmetric

$$\nabla_x J_b = B^{-1}(x - x^b)$$
$$\nabla_x J_o = H^T R^{-1}\left[H(x-x^b) - (y^o - h(x^b))\right]$$

Note: $y^o - h(x) = y^o - h(x - x^b + x^b)$

$$\cong y^o - h(x^b) - H(x - x^b)$$

$\nabla_x J = 0$ gives $x = x^a$

$$B^{-1}(x - x^b) + H^T R^{-1} H (x - x^b) - H^T R^{-1}(y^o - h(x^b)) = 0$$

$$x^a = x^b + \left((B^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1} \right) (y^o - h(x^b))$$

can show that this is W

$$H^T R^{-1}(HBH^T + R) = H^T R^{-1} HBH^T + H^T$$

$$= (H^T R^{-1} H + B^{-1}) BH^T$$

$$\Rightarrow \quad BH^T (HBH^T + R)^{-1} = (B^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1}$$

in one-variable case: $\dfrac{\sigma_b^2 H}{H \sigma_b^2 H + \sigma_o^2} = \dfrac{H/\sigma_o^2}{1/\sigma_b^2 + H^2/\sigma_o^2}$

$\therefore$ 3DVar and OI are equivalent

3DVar solves:

$$\underbrace{(B^{-1} + H^T R^{-1} H)}_{A} \underbrace{(x - x^b)}_{\delta x} = \underbrace{H^T R^{-1}(y^o - h(x^b))}_{b}$$

$Ax = b$ is a typical linear system with solvers developed by applied mathematicians.

Incremental 3DVar:

use $\delta x = x - x^b$ as control variable

$$J(\delta x) = \frac{1}{2} \delta x^T B^{-1} \delta x$$
$$+ \frac{1}{2} (H \delta x - d^{o-b})^T R^{-1} (H \delta x - d^{o-b})$$

Linearization about x^b $\quad H \equiv \left. \frac{\partial h}{\partial x} \right|_{x = x^b}$

Outer vs. inner loop:

Outer loop:
evaluate H at x, update d^{o-b}

inner loop:
minimize $J(\delta x)$ using iterative methods
(e.g. conjugate gradient).

found solution x^a that minimize $J(\delta x)$

set $x = x^a$