"Optimal" Interpolation (Lorenc 1981)

In multi-dimension. n = size of state x
p = number of observations y^o

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{state vector} \qquad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} \quad \text{error}$$

$$\Sigma = \overline{\varepsilon \varepsilon^T} = \begin{pmatrix} \overline{\varepsilon_1^2} & \overline{\varepsilon_1 \varepsilon_2} & \cdots & \overline{\varepsilon_1 \varepsilon_n} \\ \overline{\varepsilon_2 \varepsilon_1} & \overline{\varepsilon_2^2} & \cdots & \overline{\varepsilon_2 \varepsilon_n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\varepsilon_n \varepsilon_1} & \overline{\varepsilon_n \varepsilon_2} & \cdots & \overline{\varepsilon_n^2} \end{pmatrix} \quad \text{error covariance}$$

$$y^o = \begin{pmatrix} y_1^o \\ \vdots \\ y_p^o \end{pmatrix} \quad \text{observations} \qquad H = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{p1} & H_{p2} & \cdots & H_{pn} \end{pmatrix} = \begin{pmatrix} h_1 \\ \vdots \\ h_p \end{pmatrix}$$

observation operator

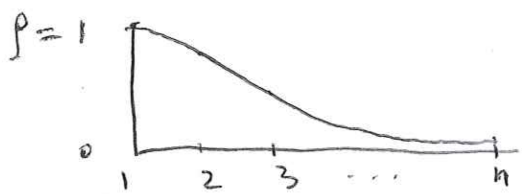
True covariance Σ is not available, use isotropic, stationary covariance B with a given correlation length scale as an assumption.

For a 1D domain case, a row in B encode the correlation function in space, the first row for example:

$$\left(\overline{\varepsilon_1^2} \;\; \overline{\varepsilon_1 \varepsilon_2} \;\; \overline{\varepsilon_1 \varepsilon_3} \cdots \overline{\varepsilon_1 \varepsilon_n} \right) = \overline{\varepsilon_1 \left(\varepsilon_1 \; \varepsilon_2 \cdots \varepsilon_n \right)}$$

$$= \sigma_1 \left(\sigma_1 \;\; \sigma_2 \rho_{12} \;\; \sigma_3 \rho_{13} \cdots \sigma_n \rho_{1n} \right)$$

$\left(1 \;\; \rho_{12} \;\; \rho_{13} \cdots \rho_{1n} \right)$ is the Auto correlation Function (ACF)



- To estimate B, one can use the "NMC" method (explained later)
- Using B in place of Σ introduces suboptimality.

Update equation for "optimal" Interpolation (OI):

$$\underset{n\times 1}{x_a} = \underset{n\times 1}{x_b} + \underset{n\times p}{W}\,(\underset{p\times 1}{y^o - h(x_b)})$$

$$\underset{n\times p}{W} = \underset{n\times n}{B}\,\underset{n\times p}{H^T}\,(\underset{p\times n}{H}\,\underset{n\times n}{B}\,\underset{n\times p}{H^T} + \underset{p\times p}{R})^{-1}$$

analysis increment: $\Delta x = x_a - x_b$

covariance between observation and state:

$$BH^T = \overline{\varepsilon_b (H\varepsilon_b)^T} = \overline{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}_b (h_1\varepsilon_b \;\; h_2\varepsilon_b \cdots h_p\varepsilon_b)}$$

variance of observation prior:

$$HBH^T = \overline{(H\varepsilon_b)(H\varepsilon_b)^T} = \overline{\begin{pmatrix} h_1\varepsilon_b \\ h_2\varepsilon_b \\ \vdots \\ h_p\varepsilon_b \end{pmatrix} (h_1\varepsilon_b \;\; h_2\varepsilon_b \cdots h_p\varepsilon_b)}$$

$$= \begin{pmatrix} \sigma^2_{y_1^b} & \cdots & \sigma_{y_1^b}\sigma_{y_p^b}\rho_{1p} \\ \vdots & \ddots & \vdots \\ \cdot & \cdots & \sigma^2_{y_p^b} \end{pmatrix}$$

where

$$\sigma_{y_i^b}\sigma_{y_j^b}\rho_{ij} = \overline{(h_i\varepsilon_b)(h_j\varepsilon_b)}$$

variance of observation: $R = \begin{pmatrix} \sigma^2_{y_1^o} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma^2_{y_p^o} \end{pmatrix}$

analysis error covariance:

$$A = (I - WH)B$$

recall in one-variable case:

$$\sigma_a^2 = \frac{\sigma_b^2 \sigma_o^2}{\sigma_b^2 + \sigma_o^2} = (1 - \underbrace{\frac{\sigma_b^2}{\sigma_b^2 + \sigma_o^2}}_{w}) \sigma_b^2$$

Problem: It is often not feasible to calculate W due to Large n and p.

Solutions: 1. Assimilate observations sequentially, assuming their errors are uncorrelated (R is diagonal)

2. Perform analysis on a subdomain of x where observations have impact:

$$(BH^T)_{ij} = \overline{\varepsilon_{b_i}(H_j \varepsilon_b)} = 0 \quad \text{if } \sqrt{i^2 + j^2} > \text{Radius of Influence (ROI)}$$

Sequential Algorithm:

For $i = 1, 2, \cdots, p$

$$\sigma_{yb}^2 = \overline{(h_i \varepsilon_b)^2} \qquad \sigma_{yo}^2 = R_{ii}$$

$$W = Bh_i^T (\sigma_{yb}^2 + \sigma_{yo}^2)^{-1}$$

$$x_a = x_b + W(y_i^o - h_i x_b)$$

$$A = (I - Wh_i)B$$

let $x_b = x_a$, $B = A$

end

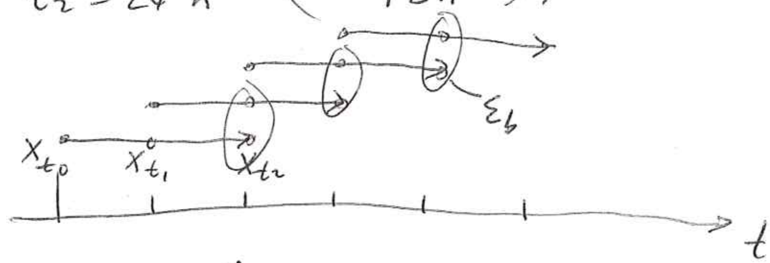
"NMC" method: (Parrish and Derber, 1992)

$$B \approx \alpha \overline{(X_{t_1} - X_{t_2})(X_{t_1} - X_{t_2})^T} = \alpha \cdot \overline{\varepsilon_b \varepsilon_b^T}$$

Scaling factor

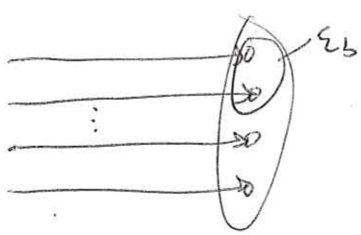
Expected value calculated from averaging many realizations

$t_1 = 48$ h, $t_2 = 24$ h (or 24 h, 12 h).



48-h forecast − 24-h forecast
is a "sample" of model forecast error

other option: using ensemble forecast



forecasts using different IC/physics/models, that characterize errors in models.

$$B = \alpha \overline{\varepsilon_b \varepsilon_b^T}$$