Observation Operator

What if the state of a system is not directly observed?

State: temperature of a stone in space, T (K)

observation: measured radiance, y (W/m^2)

observation (forward) model:

$$y = h(T) = \sigma T^4 \quad \text{(Stefan-Boltzman Law)}$$

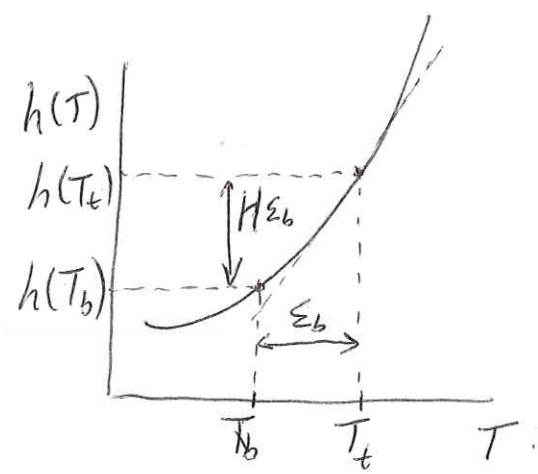
a nonlinear function

Now, we have noisy observation $y_o = h(T_t) + \varepsilon_o$

and background $T_b = T_t + \varepsilon_b$

Goal: to find best estimate T_a based on T_b and y_o

Linearization: assume T_b is close to T_t:



$$h(T_t) \cong h(T_b) + \left.\frac{\partial h}{\partial T}\right|_{T_b} (T_t - T_b)$$

$$H \equiv \left.\frac{\partial h}{\partial T}\right|_{T_b}$$ is the linearized observation (forward) operator

$$h(T_t) - h(T_b) \cong H(T_t - T_b) = -H\varepsilon_b$$

$$T_a = T_b + w\left(y_o - h(T_b)\right) \qquad (6)$$

$$T_a - T_t = T_b - T_t + w\left(\varepsilon_o + h(T_t) - h(T_b)\right)$$

$$\varepsilon_a = \varepsilon_b + w(\varepsilon_o - H\varepsilon_b) \qquad (7)$$

As before, w is determined by minimizing $\overline{\varepsilon_a^2}$

$$\frac{\partial \overline{\varepsilon_a^2}}{\partial w} = 0$$

$$\overline{\varepsilon_a^2} = \overline{\varepsilon_b^2} + \overline{w^2(\varepsilon_o - H\varepsilon_b)^2} + \overline{2\varepsilon_b w(\varepsilon_o - H\varepsilon_b)}$$

$$= \sigma_b^2 + w^2\left(\overline{\varepsilon_o^2} + H^2\overline{\varepsilon_b^2} - 2H\cancel{\overline{\varepsilon_o\varepsilon_b}}\right) + 2w\cancel{\overline{\varepsilon_b\varepsilon_o}} - 2wH\overline{\varepsilon_b^2}$$

$$= \sigma_b^2 + w^2(\sigma_o^2 + H^2\sigma_b^2) - 2wH\sigma_b^2$$

$$\frac{\partial \overline{\varepsilon_a^2}}{\partial w} = 2w(\sigma_o^2 + H^2\sigma_b^2) - 2H\sigma_b^2 = 0$$

$$w = \sigma_b^2 H\left(\sigma_o^2 + H\sigma_b^2 H\right)^{-1} \qquad (8)$$

H accounts for change in units, scaling increments in observation space to state space based on sensitivity (slope)

→ trouble when ε_b is too large ; nonlinearity

$$\sigma_a^2 = (1 - wH)\sigma_b^2 \quad \text{analysis error} \qquad (9)$$

or
$$\frac{1}{\sigma_a^2} = \frac{1}{\sigma_b^2} + \frac{H^2}{\sigma_o^2} \quad \text{analysis precision} \qquad (10)$$

scaled weight wH is between 0 and 1

if $\sigma_o^2 >> \sigma_b^2 H^2 \longrightarrow T_a \approx T_b$

if $\sigma_o^2 << \sigma_b^2 H^2 \longrightarrow T_a \approx T_b + \frac{1}{H}\left(h(T_t) - h(T_b)\right)$

Spatial Interpolation:

state variable x_1, x_2, e.g. T at Pittsburg, NewYork

observation y^o, e.g. T at State College

X_1 y^o X_2

coordinates l_1 l_0 l_2

observation operator is a linear interpolation in this case:

$$y = Hx = \alpha x_1 + (1-\alpha) x_2 \quad , \quad \alpha = \frac{l_2 - l_0}{l_2 - l_1}$$

$$= \begin{pmatrix} \alpha & 1-\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

background state mapped to observation space

$$x_b = x_t + \varepsilon_b \longrightarrow Hx_b = Hx_t + H\varepsilon_b$$

In state space:

error: $\varepsilon_b \sim N(0, \Sigma_b)$

error covariance: $\Sigma_b = \begin{pmatrix} \overline{\varepsilon_1^2} & \overline{\varepsilon_1 \varepsilon_2} \\ \overline{\varepsilon_2 \varepsilon_1} & \overline{\varepsilon_2^2} \end{pmatrix}_b = \overline{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}_b \begin{pmatrix} \varepsilon_1 & \varepsilon_2 \end{pmatrix}_b} = \overline{\varepsilon_b \varepsilon_b^T}$

In observation space:

observation prior

- innovation $y^o - (Hx_b) = y^o - (\alpha x_1 + (1-\alpha) x_2)_b \equiv d^{o-b}$

- error variance of background at observed point

$$\overline{(\alpha \varepsilon_1 + (1-\alpha) \varepsilon_2)_b^2} \equiv \sigma_{y_b}^2$$

$$= \overline{H\varepsilon_b (H\varepsilon_b)^T}$$

$$= H \overline{\varepsilon_b \varepsilon_b^T} H^T$$

$$= H \Sigma_b H^T$$

- covariance between observation prior (y_b) and background state (x_b):

$$\overline{\varepsilon_b (H\varepsilon_b)^T} = \overline{\begin{pmatrix}\varepsilon_1\\ \underbrace{\varepsilon_2}_{\text{background error of } x_2}\end{pmatrix}_b \underbrace{(\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2)_b}_{\text{background error at observed point}}} = \begin{pmatrix} \alpha\overline{\varepsilon_1^2} + (1-\alpha)\overline{\varepsilon_1\varepsilon_2} \\ \alpha\overline{\varepsilon_1\varepsilon_2} + (1-\alpha)\overline{\varepsilon_2} \end{pmatrix}_b$$

$$= \overline{\varepsilon_b\varepsilon_b^T}\, H^T$$

$$= \Sigma_b H^T$$

the update equation becomes:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_a = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_b + \overline{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}_b (\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2)_b}\, \frac{d^{o-b}}{\sigma_{y_b}^2 + \sigma_{y_o}^2}$$

in vector form: (omitting vector symbols)

$$x_a = x_b + \Sigma_b H^T \left(H\Sigma_b H^T + \sigma_{y_o}^2 \right)^{-1} (y^o - Hx_b)$$

In practice, observation operator often involve both linearization of forward model and spatial interpolation.