Least Squares Approach

Start from one-variable example, estimating temperature from two pieces of information

$$T_1 = T_t + \varepsilon_1$$

$$T_2 = T_t + \varepsilon_2$$

measurement truth error

Assumptions:

1. errors are unbiased $\overline{\varepsilon_1} = \overline{\varepsilon_2} = 0$
2. errors have known variance $\overline{\varepsilon_1^2} = \sigma_1^2$ $\overline{\varepsilon_2^2} = \sigma_2^2$
3. errors are uncorrelated $\overline{\varepsilon_1 \varepsilon_2} = 0$

→ ε_1 is a random draw from normal distribution

$$\varepsilon_1 \sim N(0, \sigma_1^2)$$

→ Gaussian random variable

What is the best estimate of T? → analysis T_a

- Linear combination of T_1, T_2: $T_a = a_1 T_1 + a_2 T_2$
  - unbiased analysis → $a_1 + a_2 = 1$
  - If $\sigma_1 = \sigma_2$ (measurements from same instrument) we trust T_1, T_2 equally: $T_a = \frac{1}{2}(T_1 + T_2)$

  but what if $\sigma_1 \neq \sigma_2$?

Least squares approach (Gauss):

find a_i so that analysis error is minimum

$$\overline{\varepsilon_a^2} = \overline{(T_a - T_t)^2} \qquad (1)$$

$$= \overline{\left(a_1T_1 + a_2T_2 - (a_1+a_2)T_t\right)^2}$$

$$= \overline{(a_1\varepsilon_1 + a_2\varepsilon_2)^2}$$

$$= \overline{a_1^2\varepsilon_1^2 + a_2^2\varepsilon_2^2 + 2\cancel{a_1a_2\varepsilon_1\varepsilon_2}}$$

$$= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 \qquad a_2 = 1 - a_1$$

$\overline{\varepsilon_a^2}$ reaches minimum when

$$\frac{\partial\overline{\varepsilon_a^2}}{\partial a_1} = 0 \;\rightarrow\; 2a_1\sigma_1^2 - 2(1-a_1)\sigma_2^2 = 0$$

$$a_1(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$

$$a_1 = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \qquad a_2 = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} \qquad (2)$$

- weight of T_1 scales with the variance of T_2.
- $a_1 = \dfrac{1/\sigma_1^2}{1/\sigma_1^2 + 1/\sigma_2^2}$, weight of T_1 is proportional to its accuracy (precision).

Now, let T_1 be from forecast (background, first guess) T_b

let T_2 be from observation T_o

Analysis $T_a = T_b + w(T_o - T_b)$ — weighted observational increment "innovation"

$$w = \frac{\sigma_b^2}{\sigma_o^2 + \sigma_b^2} \qquad (3)$$

→ Best Linear Unbiased Estimate (BLUE)

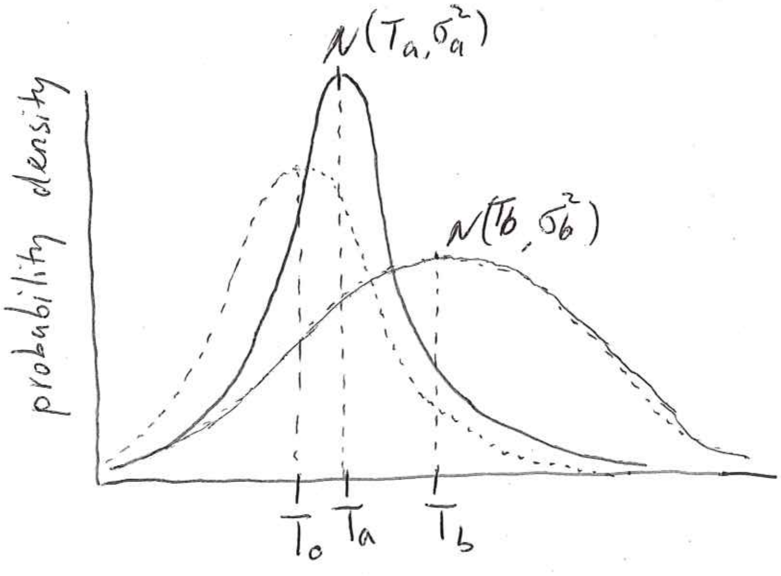
If $\sigma_o \gg \sigma_b$, $W \approx 0$, observation has almost no impact

$\sigma_b \gg \sigma_o$, $W \approx 1$, $T_a \approx T_o$, analysis fits closely to observation

variance of analysis

$$\sigma_a^2 = \frac{\sigma_o^2 \sigma_b^2}{\sigma_o^2 + \sigma_b^2} \quad \text{or} \quad \frac{1}{\sigma_a^2} = \frac{1}{\sigma_o^2} + \frac{1}{\sigma_b^2} \qquad (4)$$

$\rightarrow$ $\sigma_a^2 < \sigma_b^2, \sigma_o^2$ analysis is more accurate than background / observation



$\sigma_o^2 < \sigma_b^2$

$\rightarrow$ If $\sigma_o^2 > \sigma_b^2$, less accurate observation than background can still improve the background!

Multivariate : Two-variable example

observe wind v to constrain v, T

background error $\begin{pmatrix} \varepsilon_{Tb} \\ \varepsilon_{Vb} \end{pmatrix} \sim N(0, \Sigma_b)$ multivariate normal distribution

$$\underbrace{\Sigma_b}_{\text{error covariance}} = \begin{pmatrix} \overline{\varepsilon_{T_b}^2} & \overline{\varepsilon_{T_b}\varepsilon_{V_b}} \\ \overline{\varepsilon_{T_b}\varepsilon_{V_b}} & \overline{\varepsilon_{V_b}^2} \end{pmatrix}$$

correlation between T, V

$$\overline{\varepsilon_{T_b}\varepsilon_{V_b}} = \sigma_{T_b}\sigma_{V_b}\rho_{T_bV_b}$$

$$\begin{pmatrix} T_a \\ V_a \end{pmatrix} = \begin{pmatrix} T_b \\ V_b \end{pmatrix} + \begin{pmatrix} \sigma_{T_b}\sigma_{V_b}\rho_{T_bV_b} \\ \sigma_{V_b}^2 \end{pmatrix} \frac{(V_o - V_b)}{\sigma_{V_b}^2 + \sigma_{V_o}^2} \qquad (5)$$

- ρ_{TV} detemines how much information we can get for T from v observations

Joint probability density function for v, T